Students' alternative frameworks about the notion of limit

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Introduction

A common exercise posed about limits of functions aims at proving that the sequence of real functions, nx^2, $n \in \mathbf{N}$, does not have a limit. Frequently, students intuitively think that the limit of those functions is the positive y-axis (see figure 1). The standard reply to this answer is that the positive y-axis does not represent the graph of any function and the definition requires the limit to be a function. However one can ask, is the students' intuition wrong?

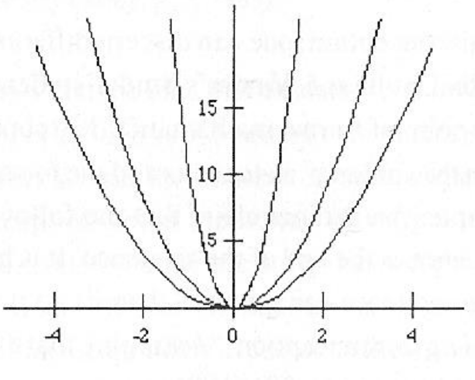

Figure 1. nx^2 for $n = 2, 3, 10$.

From the formal mathematical perspective the student's intuition leads him to an answer that is not correct. But, again, is the student's intuition wrong? Isn't there a sense in which the successive graphs of the functions infinitely approach the y-axis? Why should this intuition be labeled as incorrect? Independently of one's view about the legitimacy of the student's intuition of limit, something else occurs if the student's intuition is rejected: the students' intuition is shut off for the sake of the rigidity of the definition. Erlwanger's study (1975) supports the claim that when students develop the idea that mathematics is a big set of facts and rules, the consequences are that their mathematical intuitions do not develop and passivity relative to the accomplishment of meaningless tasks is developed.

Lakatos (1976) claims that in the process of doing mathematics nothing is taken for granted, that is, no definition is unchangeable and every theorem is subject to refutation. The imaginary classroom situation that Lakatos uses to illustrate his point can be inspirational for a real classroom situation where students contest, discuss and transform what is to be learned. But teaching in this way has to involve working with students intuitions, which have to have an important role in the learning-teaching process.

In this study[1] three students were asked to solve a geometric problem where, implicitly, several forms of the notion of limit occur. The objective of this study was to unravel students' alternative conceptions of the notion of limit. In the next section I will start by explaining briefly the difference between alternative framework and misconception, and how the notion of alternative framework was used to orient, in this study, the interpretation of students' ideas about the notion of limit.

Misconception and alternative framework

Starting by the term misconception, one can discern different uses for this term. For example, in the 1986 Davis and Vinner's study, students' statements were interpreted from the view point of formal mathematics. Misconceptions were found in the differences between the students' statements and the formal statements for the notion of limit. For example, the authors claim that the following students' statement: "The limit of a sequence is the end of the sequence. It is how far the sequence goes on. It is: as far as the sequence can go, until there is no more or it is not able" contains at least the following misconception: "Assuming that the sequence has a last term a sort of a_∞ (Davis and Vinner, p. 294, 1986).

Strike (1983) views a misconception as something important and deeply rooted in the students' mind, a misconception going beyond a simple comparison with the scientific notion. Strikes defines misconception as "an assumption that is structurally important in the student's belief system. It is something that generates mistakes. It is a piece of the student's conceptual ecology that serves to select or reject other ideas or to render them more or less intelligible" (Strike, p. 73, 1983). Let us exemplify: "a child had drawn a picture of astronauts going from the earth to the moon. He showed the rocket going around the moon and landing on the other side which he represented somehow flattened" (Strike, p. 72, 1983). One can consider that the assumption "the moon is flat" is a misconception. However, at a deeper level the misconception may be "assuming an absolute up and down". Strike (1983) views a misconception in a different than Davis and Vinner (1986) do: Strike tries to go beyond the students' statements making sense of them in a larger context.

The term *alternative framework* was introduced by Driver and Easley (1978) and it gives one more step in the direction of understanding students conceptions as legitimate theories. Alternative frameworks are "students' attempts to explain events and to abstract commonalties between those events" (Driver and Easley, p. 62, 1978). Here is an example: in a science class where students were weighing objects with a spring balance one of the students was making experiences by weighing the same object at different heights. When asked about his experiments he confessed he was surprised that the object weight did not change and explained:

> this is further up and gravity is pulling it down harder the further away. The higher up it gets the more the effect of gravity will be on it because if you just stood over there and somebody dropped a pebble on him, it would just sting him, it wouldn't hurt him. But if I dropped it from an airplane it would be accelerating faster and faster and when it hit someone on the head it would kill him (Driver and Easley, p. 2, 1985).

This statement shows that the idea that objects weigh more when are at an higher distance from the floor is an idea supported by the student's experiences, and needs to be considered correct within the context of those experiences. Saying that the student has a misconception about the notion of weight is skipping an enormous quantity of factors that constitute the meaning that weight has for the student.

Hills (1989), elaborates further on this notion of alternative framework and proposes that students' ideas should be accessed in their own terms rather than taking the scientific view as the *yardstick* to access students ideas. Instead, he suggests that, "children come to school already equipped with a more or less sophisticated understanding of how the world works" and that "this stock of understandings gradually picked up from earliest childhood may be referred to simply as common sense, or as common sense understanding" (Hills, p. 169, 1989). Hills proposes an epistemological status for students' conceptions.

What distinguishes an alternative framework from a misconception, in the present study? To find an alternative framework one has to be able to have several grounds to assess students' ideas. It is crucial to keep in mind that one is trying to figure how the framework works for the student, not for the researcher, or in comparison with the contemporary disciplined-based view.

Alternative conception and epistemological obstacle

The notion of *epistemological obstacle* is connected with the vision that mathematics is a cultural system, the elements of the system being divided into three levels: the formal, the informal, and the technical levels. On the formal level we can find elements like beliefs and judgments about what is mathematics and which are the methods that are acceptable for mathematical proof; ideas about the way mathematics evolves, for example if by accumulation or proof-refutation; philosophical positions and cultural intuitions. On the informal level we can find the implicit knowledge that allows mathematicians to put and solve problems. On the technical level we find mathematical theories: the verbalized and validated knowledge. Sierpinska (1989) claims that some of the elements at the formal and informal levels might be sources of epistemological obstacles.

Thus, a thought scheme or a rule that a person has might work like an epistemological obstacle in a certain moment of the evolution of the person's mathematical culture, solely when that person does not have conscience of the limitation of that scheme or that philosophy, and if the attitude of the individual is one of believing in the philosophy underlying the scheme thought, instead of making a choice and being aware of the consequences of that choice.

Let us exemplify how this notion works. In Sierpinska (1987) one of the students interviewed said: "1 is an approximation of 0.9(9) ... and there is an agreement upon identifying the two numbers without accepting it to be a truth" (Sierpinska, p. 393, 1987). This student shows (not only through this phrase but through extensive discussion) a finitist attitude (because, to him, bounded infinity does not exist) but at the same time a vision of knowledge that is discursive empiricist.

Students' conceptions of limits

The studies about students' conceptions of limits that are part of the literature review for this study are: Tall and Vinner (1981), Cornu (1981), Williams (1991) and Sierpinska (1987).

In the 1981 Tall and Vinner's study, the researchers face the students ideas from the point of view of formal mathematics in the sense that the students conceptions are compared with the formal definitions for limit. Working with the *concept image-concept definition-formal concept definition* terminology (1981, pp. 152-153) they claim the existence of a link between the informal and dynamic way of teaching the notion of limit (limit of function at a point and limit of sequence) and the weak concept definition for the concept in question. The concept image, on the other hand, is strong and based to some extent on the informal and dynamic ideas portrayed in the English classrooms and textbooks. From questionnaires administrated to students just arrived to the university, the authors were able to classify the concept image of the students relative to the notion limit of a function at a point in the following ways: 4 gave a *correctformal* definition, 14 a *incorrectformal* definition, 27 a *correct dynamic* definition and 4 an *incorrect dynamic* definition.

In relation to the concept limit of a sequence the authors could identify several concept images which conflict with the concept definition. Between those concept images we can find the following: — 0.9(9) is less than one because the difference between it and 1 is infinitely small; — a sequence converging to a number means that gets close to that number as n gets larger, but does not actually reaches that number until infinity; — the limit can never be the same as the sequence.

Will a simple change in non-formal terminology used in classrooms and books solve the problems students have with the formal definitions? Formal definitions are only the final product, they express a mathematical notion in skeleton form. Behind the making of a formal definition is the experience, that is, the personal constructions in which the intuition has played a crucial role. Also, in most cases the intuitive notion is more productive because it allows people to see the mathematical object from different perspectives and, consequently, to explore it more freely. The formal mathematical objects are not part of students' experience, and to be able to gain experience with them students should develop personal intuitions about those objects. However, it might take more time that what is available to the student for him to be able to reach a good grasp of the formal definition. It might also be possible that the informal and dynamic ideas are never abandoned by the person, even if the person reaches competency using the formal definition. Cornu (1981) claims it so.

Cornu (1981) reports that two models of limit emerged from the analysis of students conceptions before and after they are taught formally about limits. The models are *spontaneous models* and *modèles propres* of limits. Cornu claims that spontaneous models of limits are present in the *modèles propres*, even in students with high levels of mathematical understanding. Cornu discerned four spontaneous models of limit when searching for the meaning of the words "limit" and "tends to" with students who had not been yet exposed to any formal teaching about limits. Students view limit as a bound or landmark (also an inferior or superior bound) and limit as something that can or cannot be attained. Four models for the expression *tends to* were discerned in this 1981 study: "gets closer to, but may be far from" (going from 1 to 3 gets you closer to 10); "gets closer and is attained" (if x increases from 1 to 3 then $1+x$ gets closer to 4); "gets closer but is never attained" (tends to 0 when x tends to infinity); and, "is neighbor of" (2.8 tends to 3). In general, limit and *tends to* are used in different contexts, for example, the sequence: 0.9, 0.99, 0.999, 0.9999, etc. has limit 1 but tends to 0.9 (9). The term limit is used for sequences where its limit is attained. Cornu notes that the XVIII century mathematicians had models of limit similar to the students in his study. He suggests that looking at the historical evolution of the notion of limit may help to understand the difficulties that students have with the formal definition of limit.

The suggestion that mathematical definitions do not substitute spontaneous models for the notion of limit is reinforced by Sierpinska (1987) and Williams (1991), who claim that students' ideas are very powerful frameworks for themselves. Let us look first at Sierpinska's 1987 study.

Sierpinska (1987) reports about 4 experimental sessions with the participation of humanities students (17 and 16 year old students from a Warsaw High School) being the objective of the sessions, to confront students with their notions so that they would realize about the limitation of their conceptions. This study is part of a wider project aiming at the elaboration of didactic situations that might help students to overcome epistemological obstacles related with limits. Sierpinska's objectives seem to be conceptual change towards the technical concept of limit.

Sierpinska starts by establishing a list of epistemological obstacles related to limits with base in historical and experimental research (Sierpinska, 1985a). The obstacles listed are: the heuristic static geometrical, the heuristic static numerical, the heuristic kinetic geometrical, the heuristic kinetic geometrical, the rigouristic-

Eudox and the rigouristic-Fermat (Sierpinska, p. 373, 1987). Sierpinska also establishes a relation between those obstacles and their possible sources, namely, the notions of scientific knowledge, infinity, function and real number. In this 1987 study she explains students' conceptions of limit in connection with their conceptions relative to the nature of number and infinity, and the perspective of mathematical knowledge that students have.

During the four sessions Sierpinska identified two kinds of attitudes towards the nature of the number, infinitist and finitist, and two attitudes relatively to the nature of mathematics, the discursive and the intuitive, which are connected to seven beliefs about the nature of infinity which are connected with 8 models of limit. By the end of the four sessions Sierpinska could verify some changes in the ideas of the students, but concluded that none of the epistemological obstacles were overcome. She also claims that some mental conflicts were born and maybe that constitutes a starting point for further research.

Williams (1991) concluded that students have a very strong conceptual knowledge relative to limits. Students separate procedural knowledge from conceptual knowledge, being more convinced and attracted to the "ease and practicality of a model" which they consider more important than the formal view of the concept. Counterexamples seem not to have the shocking effect that teachers and researchers expect. Students considered counterexamples as minor exceptions, not a reason for abandoning personal constructs.

Williams' report leads one to think that counterexamples do not in general destroy a student's personal construction. Students put a great deal of work into their personal construction and work hard to extend them to new examples instead of throw them away. Therefore we should not avoid students' constructions or avoid metaphoric thinking and teach in a strictly formal way, but to help students to be critical and implement the habit of testing their constructions.

Procedures of this study

This study was done in cooperation with three students. One of the students majored in math and the other two students had taken one calculus course. Each of the students was interviewed once and individually. In the beginning of each interview were given to the student a drawing of the circle of center (0 , 0) and radius 1 and some lines of the sequence of lines that go through (1 , 0) and the point $1-\frac{1}{2^n}$;

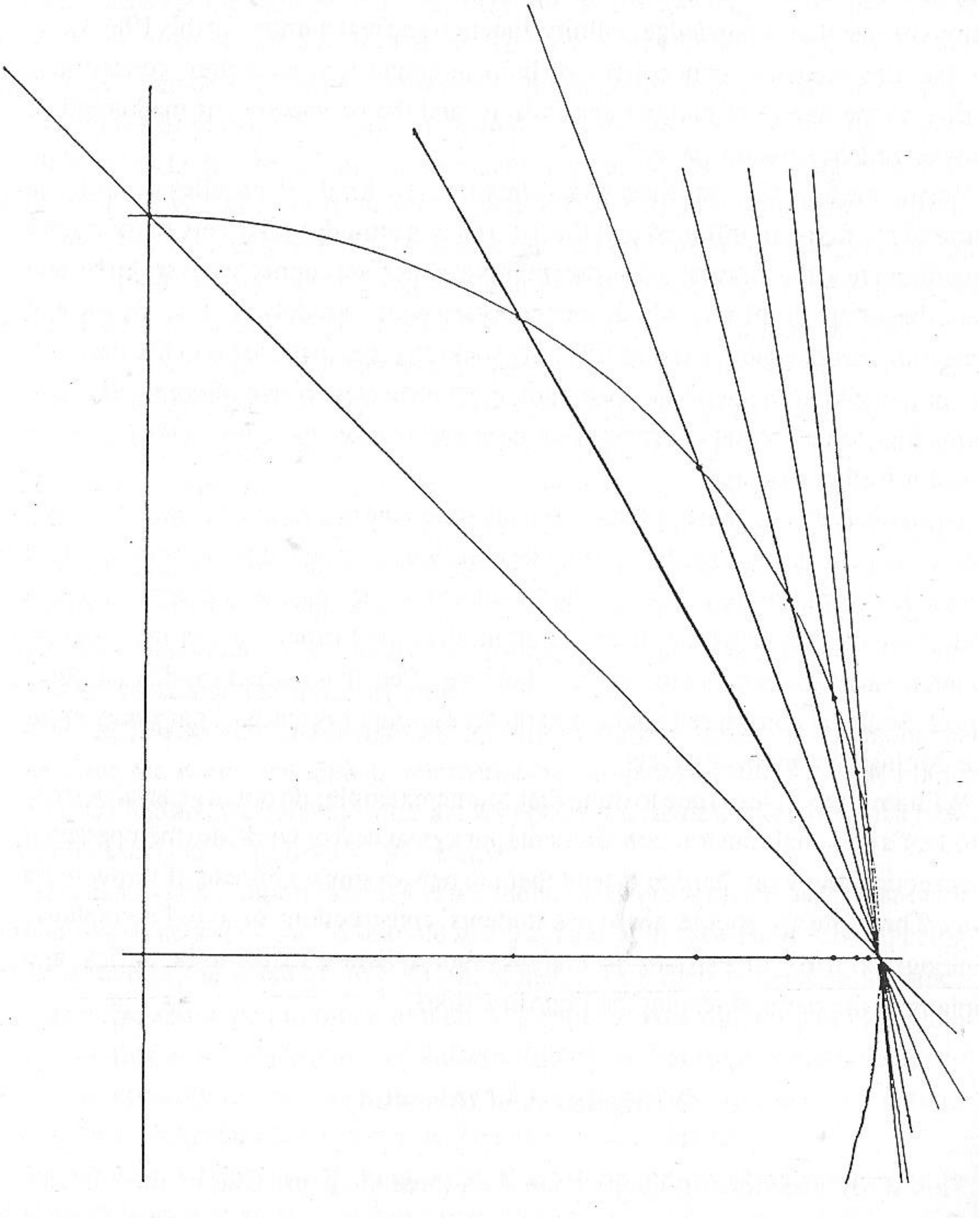

Figure 2. What is the limit of the lines?

and in a separate sheet the problem was stated, in the following way: "Given the circle of radius one, for each n positive integer, consider the straight line that goes through (1 , 0) and the point in the circle with x-value $1-\frac{1}{2^n}$. What is the limit of the lines?" (see figure 2).

The expected answer is that the limit is the vertical line through (1 , 0). The students are then asked to reflect over the reasons of their answer, and their notions of limit became explicit through their justifications. Students ideas of limits were interpreted using the underlying methodology of alternative framework discussed before. To solve the problem they were required to use the program Function Probe© (Confrey, 1991). The students' voices were audio taped and their interaction with the Probe was videotaped. Function Probe© can be described has a program with multirepresentational features namely: Table, Graph and Calculator. A student can travel between representations, for example, work done in the table representation can be transferred to the graph representation.

Analysis and findings

All three students used the Graph and Table features of Function Probe. They started by drawing the circle and next they computed by hand the equation of the first line. Then they used the Table feature to compute slopes in series. The tables' top row of each student's table are represented below. During the interview the students increased several times the range of n as well as the results' precision.

Table 1. Tables' top row of each student's table.

Jed	n	$x = 1-\frac{1}{2^n}$	$y = \sqrt{1-x^2}$	$m = \frac{0-y}{1-x}$	$b = -m$	$c = 2 \times b$
Telma	n	$x = 1-\frac{1}{2^n}$	$y = \sqrt{1-x^2}$	$z = \frac{y}{x-1}$	$w = x-1$	
Marvin	n	$a = 1-\frac{1}{2^n}$	$s = \sqrt{1-x^2}$	$m = -\frac{s}{1-a}$	$b = \frac{s}{1-a}$	

The first alternative framework described here is an alternative framework of the notion of convergence. Students see a sequence not just as a sequence of numbers but as an object with dynamic characteristics. This can be observed, not only through the language that they used throughout the whole interview, like: "is getting closer", "goes faster than", but also when they refer to the values of the sequence $(1-\frac{1}{2^n}, 0)$ as getting to the point limit (this is, (1 , 0)) faster than the values of the sequence $(1-\frac{1}{2^n}$, correspondent point on the upper semicircle). The expression "gets faster" is not employed by the student meaning speed: a sequence going faster than another means the values are going ahead, that is, one sequence is getting to the limit sooner than the other. Like attributing to the index of the sequence some sort of time, that determines how much one sequence goes ahead of the other. For example, in the case of Jed's table when $n = 19$ the sequence of the x-values is already 1 but the sequence of the y-values only becomes 0 when $n = 37$ (see figure 3 and 4).

n	x=1-1/2^n	y=(1-x^2)^.5	m=(0-y)/(1-x)	b=-m	c=2*b
0.00000	0.00000	1.00000	-1.00000	1.00000	2.00000
1.00000	0.50000	0.86603	-1.73205	1.73205	3.46410
2.00000	0.75000	0.66144	-2.64575	2.64575	5.29150
3.00000	0.87500	0.48412	-3.87298	3.87298	7.74597
4.00000	0.93750	0.34799	-5.56776	5.56776	11.13553
5.00000	0.96875	0.24804	-7.93725	7.93725	15.87451
6.00000	0.98438	0.17608	-11.26943	11.26943	22.53886
7.00000	0.99219	0.12476	-15.96872	15.96872	31.93744
8.00000	0.99609	0.08830	-22.60531	22.60531	45.21062
9.00000	0.99805	0.06247	-31.98437	31.98437	63.96874
10.00000	0.99902	0.04418	-45.24378	45.24378	90.48757
11.00000	0.99951	0.03125	-63.99219	63.99219	127.98437
12.00000	0.99976	0.02210	-90.50414	90.50414	181.00829
13.00000	0.99988	0.01562	-127.99609	127.99609	255.99219
14.00000	0.99994	0.01105	-181.01657	181.01657	362.03315
15.00000	0.99997	0.00781	-255.99805	255.99805	511.99609
16.00000	0.99998	0.00552	-362.03729	362.03729	724.07458
17.00000	0.99999	0.00391	-511.99902	511.99902	1.02400e+3
18.00000	1.00000	0.00276	-724.07665	724.07665	1.44815e+3
19.00000	1.00000	0.00195	-1023.99951	1023.99951	2.04800e+3
20.00000	1.00000	0.00138	-1448.15434	1448.15434	2.89631e+3
21.00000	1.00000	0.00098	-2047.99976	2047.99976	4.09600e+3
22.00000	1.00000	0.00069	-2896.30920	2896.30920	5.79262e+3
23.00000	1.00000	0.00049	-4095.99988	4095.99988	8.19200e+3
24.00000	1.00000	0.00035	-5792.61867	5792.61867	1.15852e+4
25.00000	1.00000	0.00024	-8191.99994	8191.99994	1.63840e+4
26.00000	1.00000	0.00017	-11585.23746	1.15852e+4	2.31705e+4
27.00000	1.00000	0.00012	-16383.99997	1.63840e+4	3.27680e+4
28.00000	1.00000	0.00009	-23170.47498	2.31705e+4	4.63409e+4
29.00000	1.00000	0.00006	-32767.99998	3.27680e+4	6.55360e+4
30.00000	1.00000	0.00004	-46340.95000	4.63410e+4	9.26819e+4
31.00000	1.00000	0.00003	-65535.99999	6.55360e+4	1.31072e+5
32.00000	1.00000	0.00002	-92681.90002	9.26819e+4	1.85364e+5
33.00000	1.00000	0.00002	-131072.00000	1.31072e+5	2.62144e+5
34.00000	1.00000	0.00001	-185363.80005	1.85364e+5	3.70728e+5
35.00000	1.00000	0.00001	-262144.00000	2.62144e+5	5.24288e+5
36.00000	1.00000	0.00001	-370727.60009	3.70728e+5	7.41455e+5

Figure 3. Jed's table window.

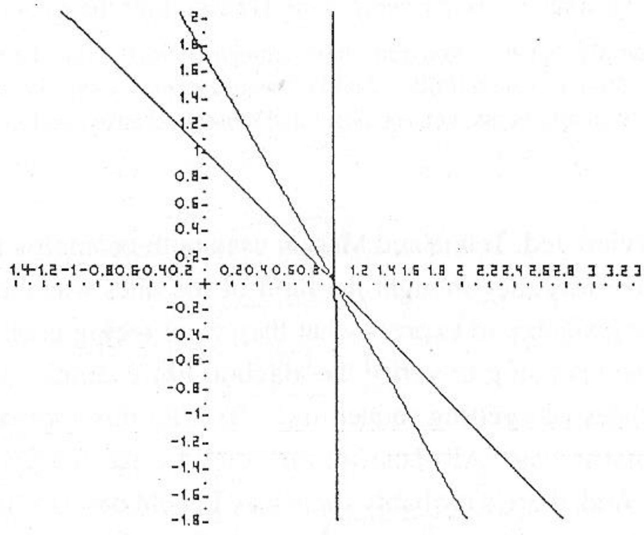

Figure 4. Jed's graph window.

So the first sequence is 16 terms sooner than the second sequence. Here is the student statement:

> So, is ah, I want to say that that's going to infinity. The slope of the line getting infinite because the *x* column is getting closer to being one. It's getting to one faster than the *y* column is getting to zero. Yeah, like if you look way back here when *n* is eighteen, the *x* value is already one but the y value is point zero zero two seven six. Like when *n* is seventeen, the one before that, then ah, the change in y is over the change in *x* is point zero zero three nine one over point zero zero zero zero one... The slope of the line is heading towards infinity...

Dynamic views were present when students try to relate the limit line with the lines of the sequence. Limit and sequence seem to be put together, by the students, through the dynamic behavior of the sequence relative to the limit point. From the three students, Marvin separated more clearly the limit from the process of convergence. He names "infinity" the process of "keep doing it" or "getting close to but never reaching". The limit is something that he calls limit because the elements in the sequence are behaving in a "convergent way" towards the limit point:

> I mean the b is where it intercepts the y axis which keeps going higher and higher and approaches infinity... and the slope will be going to negative infinity... it will just keep going like this until is almost perpendicular... almost tangent to the circle. [Pauses] It [the line] can go as close to vertical as possible, but it can't go vertical... it approaches

> the vertical line so the limit will be the vertical line. [Pauses] Like these points [1- $\frac{1}{2^n}$] will never reach one like but ah… you don't have enough decimal points here but you would just put one down or round it off… So like I would probably say you could reach the vertical line but actually is just getting closer and closer, because you have an infinite number of lines .

During the interview Jed, Telma and Marvin used both geometric and algebraic approaches to justify why they thought the limit of the lines was the vertical line through (1 , 0), using algebra to express what they were seeing geometrically and using geometry when needing to verify the algebra. For example, Jed compared which of the sequences was getting sooner to (1 , 0), after this approach he looked at the lines in a geometric way: "All of the lines are rotations around that point of other lines that are there. And, there's probably some way I could describe how much the angle increases each time". The other approach, that Jed used, was to observe how the slope was increasing. Observing the table he built in Probe, Jed predicted that the ratio between the slope of the $(n + 2)^{th}$ line, $M(n+2)$, and the slope of the n th line, $M(n)$, converges to 2. He starts then to compute, in a sheet of paper (see figure 5), figure $\frac{M(n+2)}{M(n)}$, and simplifies this expression by performing some "approximations", for example, $2^{n+1} - 1$ "is approximately" 2^{n+1}. These approximations are not done by cutting digits or round off but by means of a substitution, in $\frac{M(n+2)}{M(n)}$, of a sequence by another sequence that does not change the limit of the expression. When Jed says "$2^{n+1} - 1$ is approximately 2^{n+1}" he means that for the purpose of "taking limit" they are alike. As above, Jed sees a sequence as a whole so that he can compare two sequences in terms of how are they converging.

Telma also commuted between algebraic and geometric devices while solving the problem. She first observes that the sequences x and y (see Telma's table top in figure 5) converge to 1 and 0, respectively. She sees that the slopes depend of the two sequences and she says that the slopes are getting bigger and which means that the lines are getting vertical. After this she goes back to the sequences and she says that x-1 is converging faster to zero than y: "I mean the x-1 is getting smaller and smaller the y is too but the x-1 is getting smaller faster". She tries to make also a symmetry argument:

> Well because… well I do not know exactly but just because its a circle and its going to be… the distance between either side is going to be equal… its also going to be zero but… but the point [pauses] the closer you get the more the line goes. Its gonna be a line its gonna have the same distance on either side… more the circle… right next to it like

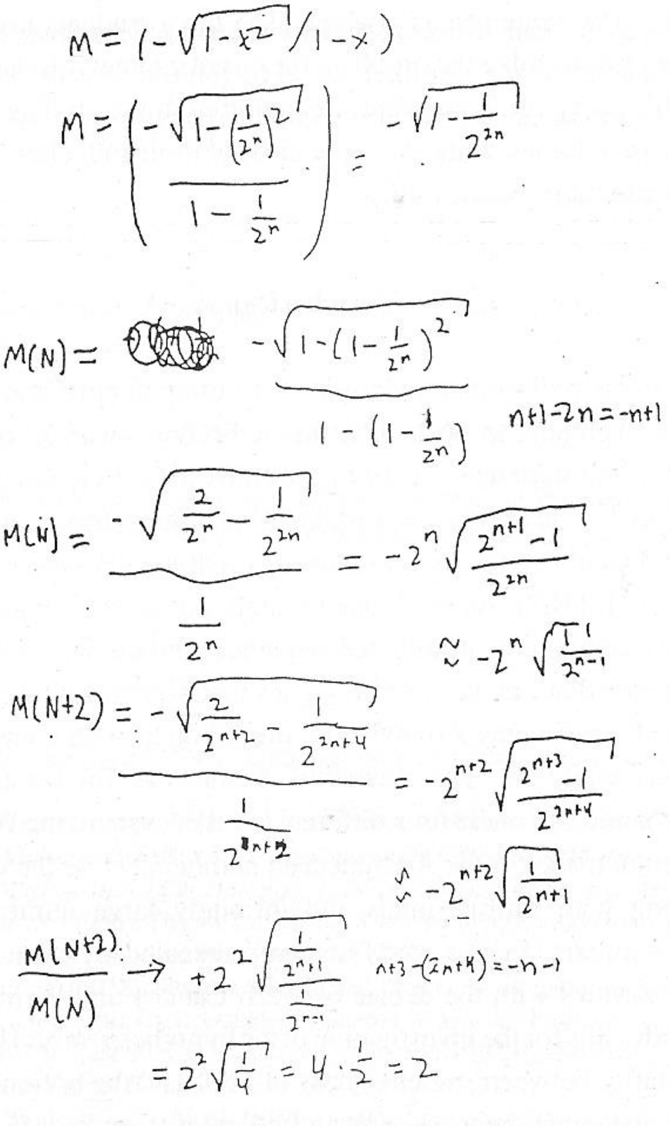

Figure 5. Jed's handwritten computations.

the limit I guess of where the circle is before you get to the x-axis before and next to it.

The three students made a distinction between sequence and limit for which the sequence converges. However, they put limit and sequence together through the dynamic behavior of the sequence as it approaches the limit point. They made comparisons between sequences in terms of their convergence what might lead us to think that they view sequence as a whole. The three students used algebraic and geometrical devices to solve the problem: they used geometry to justify algebra and vice-versa. This study confirms what was reported in the studies of the review of literature relative to the students' dynamic ideas of limit and relative to the diversity of those ideas and their practicability.

Conclusion

According to the philosophy underlying the notion of epistemological obstacle, for a student to overcome an obstacle he has to become aware of the limitations of his conceptions: "the student will have to rise above his convictions, to analyze from outside the means he had used to solve problems in order to formulate the hypotheses he had admitted tacitly so far, and become aware of the possible rival hypotheses" (Sierpinska, p. 374, 1987). We might legitimately ask what are the rival hypotheses concerning a dynamic view of limit and sequence, and the view of a sequence as a whole? Let us note that Lakatos (1978) argues that Weirstrass framework for limit is quite distinct from Cauchy's framework, providing us with a view of what these rival hypothesis might be. The Weirstrass framework for limit is static while Cauchy's is dynamic and made for a different number system: the Weirstrass notion of limit is a notion made for the Archimedean numbers, while the Cauchy's notion implies working with infinitesimals and infinitely large numbers besides the Archimedean numbers. In this study students revealed dynamic views of limit suggesting similarities with the debate between Cauchy and Weirstrass described above. This study calls for the investigation of the hypotheses raised by Cornu (1981) about the similarity between students ideas of limit and the notions of limit of the XVIII century mathematicians, in order to find out if there is in fact such parallel.

This study also suggests some directions for the classroom. Maybe, if the formal notion of limit is presented as a particular and less rich interpretation of students' notions of limit, the students can have more of a chance of understanding the utility of the formal definition. In that case, students would not be asked to refuse their intuitive dynamic notions but to look at a very clever and useful tool that can be taken out of their intuitive notions. We need to start to investigate the whys of their dynamic conceptions of limit and we need to find ways that truly integrate those ideas in the curriculum development.

The use of the methodology underlying the notion of alternative framework gives voice to different ways of looking at limits. It allowed me to look at students ideas as not defective and as having an important role in the process of solving a problem.

Notes

[1] A more detailed description of this study can be found at Moura (1993).

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ABSTRACT. The ideas of three students about the notion of limit were investigated and interpreted using the methodology underlying the notion of alternative framework. Students revealed dynamic ideas about limits and sequences, and commuted between geometric and algebraic representations to think about a geometric problem about limits. It is argued that student's intuitions about the notion of limit have a crucial role in the learning and teaching of those notions, and consequently student's intuitions should play a role in the curriculum development.