Realistic Mathematics Education — A different approach to learning and instruction

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Introduction

In this article, we contrast the top-down approach to learning and instruction of traditional instructional design models with the bottom-up approach of the *domain specific instruction theory for realistic mathematics education (RME)*.

This theory is based on the educational philosophy of Freudenthal of 'mathematics as an human activity', which has sparked several decades of *developmental research* in the Netherlands. Instruction sequences that fit his orientating philosophy have been developed, resulting on this empirically grounded instruction theory, RME.

In order to offer some background to situate RME I will sketch a brief history of theories on learning and instruction in connection with their implications for mathematics education. In doing so I will try to justify the choice for the realistic mathematics approach. The key principles of RME and the type of research that gives rise to this theory 'Developmental research' will be discussed, as well as the use of models within it, making a contrast with the traditional use of *manipulatives*.

Learning theories

What is 'learning'

Before we discuss about ways to achieve effective learning one has to know what learning is actually all about. We say that someone has learned something, when there's a small relative change in one's behaviour or a relative change in one's possibility to act differently. In other words, someone learns something when one can do something that one couldn't do before. In short, learning is a complex active process of working out information, from which results that one behaves or can behave differently from the way he did before (Boekaerts and Simon, 1993). Note that we are talking about a permanent change. What one learns, one doesn't forget easily. One can also talk about the degree of flexibility of the learning result. This has to do with its degree of transfer. When a learning result can be applied in a wide range of situations different from the ones where this learning result took place, then we say that the learning result has a high degree of transfer.

If we accept Boekaerts and Simon's idea about what learning is, it becomes clear what the goal of teaching is. Opinions diverge when one considers the question: how does one learn under a teaching situation and how should one be taught? In the next paragraph we shall give an overview of different theoretical streams in Education Psychology and some more specific learning theories.

Behaviorist theories

Within Behaviorism there are two different opinions about learning. One is the classic theory of conditioning, often related to the work of Pavlov. In general, this is about an association principle lauded as this: if two things happen to someone at the same time or during a short period behind each other it can happen that one of them gets the same reaction as the other, when it happens again to the same person. The same phenomenon can happen with animals. The classic example of Pavlov's work is the one where a dog gets food and simultaneously listens the sound of a bell for a couple of times. Later on, the dog starts salivating just by hearing the bell. The dog associates its sound with food, provoking the bell the same reaction as seeing the food.

In the other theory, associated with the work of Skinner, learning is a reaction to behaviour. According to this theory one has to first determine the behaviour to be created or changed in the person. After that, one has to choose an award. Some kind of awards work with everybody, others are specific for the person we want to teach. The idea is to award the person when he behaves the way that one wants him to.

In education psychology, this theory has dominated for a long time. One believed that letting students take small steps and frequently prizing them, applying successively approximations to the desired goal (that students show a certain behaviour- and which proves that they have learned what they were supposed to), learning would take place.

From this perspective, the mind is treated like a black box: what is inside ones head can not be known and is not even interesting. What can be controllable is what 'goes in and out' of one's head, stimulus and behaviour.

Cognitive theories

In cognitive theories, the attention of psychologists does no longer go to how instruction should be given, but their focus is on information processing. In opposition to behaviourists, cognitive psychologists' interest is exactly the black box, which is therefore no longer a black one. The object of their study is people's mind in order to explain and predict human behaviour. This theory is characterised by the conception that knowledge is stored as an organised entity of small elements of knowledge, a cognitive structure.

In cognitive education-psychology theories, the central question is how instruction can be shaped to obtain an optimal guidance of the learning processes. At first, more attention was given to the way students work out information, and to the nature of information representation in human's memory. Later on, studies were done on people's strategies on areas like memory, learning, thinking and solving problems. One was interested in understanding how students structure and organise information, how they record this in their memories and how they access this information when they need it, for instance, in order to solve a problem.

It is not believed that acquisition of knowledge happens automatically. Instead, learning is an active process: the student has to percept the relevant information, he has to structure it and subsequently connect it with what he already knows.

This theory views mathematics as a ready-made system with general applicability, and mathematics instruction as breaking up formal mathematics into learning procedures and then learning to apply them.

Theory of Ausubel. Ausubel created in the sixties his cognitive learning theory, far before cognitive psychology was in vogue (1968). The main idea in his theory is that new knowledge has to be connected with already acquired one in order to make possible what he calls meaningful learning. Otherwise we can only talk about rote learning. According to his theory, information is organised in hierarchies in the memory, such that each concept is, or may be, connected with other concepts. He refers to this as a cognitive structure.

To talk about meaningful learning it is necessary to find good bind points where new information can be attached. Ausubel talks about three ways to do this. One possibility is to connect a concept with a wider range of other related concepts in the same level of structure, *combinatorial learning*. Another possibility is to connect more abstract concepts with more concrete ones from a lower level in the structure, *superordinate learning*. Finally, a concept can be related to more general ones, in an upper level in the cognitive structure, *subordinate subsumption*.

According to Ausubel, this last one is the best way to achieve meaningful learning. When learning more concrete concepts, a general concept can have a function of integration and at the same time of explanation which makes this way of learning more adequate than the other two.

Other important idea in his theory is the idea of discriminability of information in the cognitive structure. He talks then about the principle of integrative reconciliation where he means that, at the end of a learning process, it is important to compare differences and similarities between old and new information in order to prevent it from becoming unclear later.

At the time of Ausubel, there were in the United States psychologists who defended discovery learning, where students learned through self experimenting and discovering, but Ausubel believed that exploratory learning was much more efficient than the discovery one. Furthermore, it is known from investigations that the cognitive structure is not always organised in hierarchies like Ausubel believed. Moreover, discovery learning is in certain situations efficient and effective. However his ideas of attachment or meaningful learning are still in some way present in modern theories.

Theory of Van Parreren. Van Parreren is a Dutch psychologist who developed in the sixties a cognitive learning theory called *system theory*. He stressed the importance of differentiating learning results and the activity that leaded to these results. When a student comes to the right answer of a problem it does not yet mean that he has used the correct method. The outcome can just be a coincidence.

An important concept in his theory is the concept of memory trace. According to Van Parreren, learning is a process from which a trace results in one's memory, which gives the possibility for someone to act in a certain way. Learning is thus seen as a process through which someone reaches his possibility to act. This possibility is guarantied by the traces learning leaves back in one's mind. These traces are connected to each other organised in a system, so that every time someone activates a trace using what he learned, other connected traces will also be automatically activated.

However, these systems are not all connected to each other. Van Parreren talks about the existence of a system division. According to this, information about different subject areas is saved in different places in our memory, organised in separated systems, which, on the one hand prevents confusion of ideas. On the other hand, this system division brings other problems with it. It is, for instance, more difficult to access certain information at a specific moment, when it is not connected with another already activated.

Action psychology theories

These theories are also called social-cultural theories, due to the importance they give to the social and cultural environment in the learning process. Contrary to cognitive theories, where learning is seen from the individual perspective, in action psychology theories learning is seen as a social practice.

A good example of such a theory is the cultural-historical one that Vygotsky developed in the beginning of the century, based on the processes of emergence of thinking activity, which he named mental activity. Examples of such activities are mental comparisons of concepts, construction of conclusions or mental calculations of a sum of 2 numbers. According to him, the basis of these activities are external perceptible ones, which are made internal through a social process. In other words, Vygotsky sees learning as a process where a certain external action (taken as shared by the community in that culture) is transformed in a mental activity. Dialog has in this process an important role. He differentiates four principles in the desirable dialog: the principle of interiorisation, the zone of nearest development, the central role of the adult and the principle of social-communicative source of mental activity.

The first principle refers to the sources of mental activity, which has its basis in external activities. According to the principle of the zone of nearest development, education has to be orientated to what a student still can't do alone (but almost and certainly with help of an adult). The third principle refers to the importance of the guidance of an adult to stimulate students to perform independently the tasks of their zone of nearest development. The last one stresses the importance of the dialog, not only between children and adults, but also among children.

The theory of Gal'perin. The theory of Gal'perin is included in the action psychology as he worked out theoretical starting-points of Vygotsky (Parreren & Carpay, 1980). His theory stresses the stepwise formation of well-formed mental actions. For Gal'perin, higher mental activities are developed through a process of orientation. Therefore, he suggests a stepwise instruction plan in order to guide students in learning or changing a certain mental action. He differentiates different steps in this plan, which was used as heuristics in instruction design: the first step or orientation phase concerns the analyses of what is ought to be learned and in which sequence. It follows the phase of carrying out material activities. In order to be possible for someone to learn a new action, one has to perform it. Students work with external perceptive activities and they get feedback about their work. The third phase is the verbalisation of the student's actions that took place in the phase before. Through this process, students are stimulated to generalise and abstract, and the external action can be transformed into an internal one. In the fourth phase, students work more and more independently till they reach the last stage of the learning process, achieving an high mental level.

Constructivism

Constructivist theories have their roots in the eighties, through the contribution of investigators as Von Glaserfeld, Steffe and Cobb, as they brought radical constructivism under attention of mathematics education investigators (see, for instance, Kilpatrick, 1987). The radical constructivism of Von Glaserfeld has its roots in epistemology, cognitive psychology and cybernetic (Gravemeijer, 1995). According to him, learning has to do with self-organisation, which means that learning is an activity that takes form according to the intention of the person. Or as Cobb puts it:

> (…) learning is characterised as a process of self-organisation in which the subject organises his or her activity to eliminate perturbations (Cobb, 1994).

According to constructivism, it is not possible for human beings to know an objective reality. In other words, reality as we know it is dependent of our way of knowing it.

> We construct our knowledge of our world from our perceptions and experiences, which are themselves mediated through our previous knowledge (Simon, 1995, p. 115).

Learning is then, in this perspective, the process through which a person adapts to his experiential world. The issue is not to know whether something is true or not, but rather to know if it fits our experiential world. Von Glaserfield talks about "viability". A concept is then viable, or it fits our reality, if it makes sense to us at that moment and fits our goals and purposes. This is also applicable to the theory of constructivism itself, and we can therefore say that constructivism is only true as it helps us making sense of our experiences, as Confrey (1995) mentions.

Whenever our experience differs from what we know (our reality), a process takes place in order to re-establish this disequilibrium, the learning process. In this theory, the idea that learning is an active, constructive process is even stronger than in the cognitive and meta-cognitive theories. People in general, thus students as well, create successive new internal representations of information using all their previous knowledge. Note that this knowledge differs from person to person, once everyone experiences different things, which makes impossible to reproduce one's process of learning. Each person experiences a unique process of learning. Learning is thus for constructivists the construction of new internal representations of information, using already available internal representations. An internal representation that doesn't fit the general accepted as true representation, is called *misconception* or like constructivists prefer to call *alternative conception* (this last name is more adequate for the constructivists idea that reality is subjective).

Paris and Byrne (Boekaerts *et al.*, 1993) try to grasp what is common in the wide variety of conceptions of constructivism by listing six fundamental ideas in constructivist theories:

- Students have an intrinsic motivation to search for information
- The process of understanding is not simply a process of registering information given; it goes beyond that. Students bring structure to it, they (re)organize and they generalize the information given.
- There are two ways for changing mental representations: through new experiences or simply as a result of spontaneous development.
- There are infinitive levels of understanding. Constructivists believe that by an internal reorganisation of mental representations and by reflection, refined understanding is reached.
- Someone's level of knowledge and experience determines what he at that certain moment can learn. The amount of guidance a student needs, his capacity to process information and the kind of tasks he finds difficult, various with his age.
- Reflection and reconstruction stimulate learning. People reflect by nature about their own behaviour. When students reflect about the way they learn, what

and how they learn, they construct themselves a mental theory about learning.

One can also wonder if knowledge development is essentially a social or a cognitive process. Radical constructivists prefer a cognitive or psychological perspective, where knowledge development is seen as individual construction. Social interaction is taken into account but the major attention goes to the resulting reorganisation of individual cognition. Other constructivists see higher mental processes as socially determined. 'Knowledge resides in the culture, which is a system that is greater than the sum of its parts' (Simon, 1995).

Radical constructivists believe that the world out there is filtered by our perception and, therefore, it may even not exist. When we take this into account, and considering a common sense idea that the way one perceives his world is influenced by his culture, then I must agree that knowledge development is essentially a social-process. However, if we apply the same idea to this conclusion (that what we belief to be real is mediated by our perception), we see that we can discuss long about this. It seams then wiser to analyze what can we learn out of this two perspectives, instead of trying to defend the one or the other.

According to this:

> It is useful to see mathematics as both cognitive activity constrained by social and cultural processes, and as a social and cultural phenomenon that is constituted by a community of actively cognising individuals (Wood, Cobb, & Yackel, 1995).

More recently, the idea that is brought to the fore is that knowledge and capacities are in one's mind connected with the specific context where these were experienced. 'Each experience with an idea-and the environment of which that idea is a part – becomes part of that idea' (Duffy & Jonassen, 1991, p. 8). According to proponents of this notion of 'situated cognition' knowledge and capacities are connected with the situation where they were obtained and therefore transfer doesn't just happen automatically from context to context.

Out of the previous, we get an idea of the diversity of theories within constructivism. They all share the central idea that "students construct their own knowledge" but they vary in what they mean by this adagio.

Socio-constructivism of Cobb, Yackel and Wood. Socio-constructivism as advocated by Cobb, Yackel and Wood integrates social interactive theories with constructivism, solving a problem created by radical-constructivists. As we have seen, according to these, there's no way to ensure that there is a reality "out there". One has just access to a self-made model of a reality based on one's experiences. This brings up a problem to mathematics education. According to this idea, both conceptions of students and of teachers or experts have equal value and one has therefore no arguments to stipulate a direction for teaching.

Cobb, Yackel and Wood solve this problem by accepting the general assumed knowledge about reality as point-of-departure: 'the *taken-as-shared* mathematical interpretations, meanings, and practices institutionalised by wider society' (Cobb, Yackel and Wood 1992, p. 16). The goal of instruction is then that students "learn" this taken-as-shared knowledge. According to Cobb, learning mathematics is to make yours the ideas and ways of working of mathematics community (Gravemeijer, 1995). The classroom is seen as a community where its own mathematics is developed, as it develops its own "taken-as-share" meanings, interpretations and practices.

The role of the teacher is then to guide students in the self-construction process they go through (the learning process), in order to diminish the gap between the taken-as-shared knowledge of the classroom community and the one of wider society. With self-responsibility and intellectual autonomy of each member of the community it is attempted to achieve together mathematical knowledge.

> More generally it is by capitalising on students' mathematical activity that the teacher initiates and guides the classroom community's development of taken-as-shared ways of mathematical knowing that are compatible with those of the wider community (Cobb *et al.*, 1992).

In this process an important role is played by what is called '*classroom social norms*'. They include what is understood within a classroom community as being effective participation, the expectations of the teacher and the students about each other's responsibilities, the conception of what it means to do mathematics or the ways mathematical validity is established. This way, it becomes clear for the students what is expected from them in the mathematics class. By establishing the corresponding social norms, students will, for instance, feel more responsible for the correctness of their own answers. In a similar manner, norms about mathematical progress can help the teacher to give the learning process of the classroom community direction without injuring the students' intellectual autonomy.

In social interaction, teacher and student try to come to shared understanding by adjusting their interpretations and reactions to each other's reaction in a process of 'negotiation of meaning'.

During communication, a lot of information is taken as shared by both individuals of the process, and therefore misunderstandings are natural. This is easy to understand from a constructivist perspective. What happens is that both parts construct, in fact, a model of what at that moment is believed to be taken-as-shared, according to ones' perception. Once this is just a model of the real situation, misconceptions are natural to occur. This process of negotiation is then no more than the process through which each other's models of what is taken-as-shared are adapted to current believes.

According to this theory, the goal of teaching mathematics is to be able to participate in the practice of the mathematical community (Gravemeijer, 1992). The process that leads to this is referred to as "acculturation", since learning mathematics is seen as integrating mathematics "culture" in one's own culture.

Realistic Mathematics Education (RME)

Realistic Mathematics Education is grounded in Freudenthal's (1973, 1991) interpretation of "mathematics as an human activity". According to him, students should be given the opportunity to reinvent mathematics by mathematising either mathematical matter or subject matter from reality. Important about the subject to be mathematised is that students experience it as real. This is one of the reasons this kind of education is called Realistic.

According to Freudenthal, mathematising is a key process in mathematics for two reasons. First of all, not only mathematising is the major activity of mathematicians, but also it familiarises students with a mathematical approach to everyday life situations. The second reason has to do with an important concept in this theory, the principle of *guided reinvention*. The final step of mathematicians while developing mathematics, that is formalisation through axiomatising, is the learning starting point in traditional mathematics education. This is according to Freudenthal anti-didactical once the way mathematicians came to their conclusions is here turned upside down. According to him, mathematics education must be organised as a process of guided reinvention, where, through mathematising, students can experience to some extent a similar process to the process through which mathematics was invented.

Note that Freudenthal uses the word mathematising in a broad sense, meaning the process of not only recasting an everyday problem situation in mathematical terms but also a process within mathematics. In his view, the goal for mathematics education should be to support a process of guided reinvention in which students can participate in learning processes that parallel, to some extent, the deliberations surrounding the historical development of mathematics itself (Gravemeijer, Cobb, Bowers and Whitenack, in press).

So far, we have presented, in short, Freudenthal's education philosophy. This has sparked several decades of developmental research (Gravemeijer, 1994a) and is still guiding current work, where instruction sequences that fit Freudenthal's orientating philosophy are developed. The result is an instructional theory in progress, Realistic Mathematics Education (RME), which has been developed by reconstructing the local theories implicit in these instructional sequences. As a consequence of this, RME consists of interrelated series of instructional sequences, local instruction theories and more general principles. From this last, we can distinguish three key ones, which can be seen as heuristics for instructional design: *guided reinvention and progressive mathematising*, *didactical phenomenology* and *emergent models*.

Key principles of RME

Guided reinvention and progressive mathematising. As mentioned before, according to the principle of guided reinvention and progressive mathematising, students should experience a learning process similar to the one followed by mathematicians. An instruction route has to be found that allows students to discover the intended mathematics by themselves. The developer has to imagine a route through which he could have arrived to the outcome himself. In order to do this, two sources of inspiration can be used.

One of them is the history of mathematics. Knowing how mathematicians came to their conclusions can help the developer to lay out intermediate steps through which the intended mathematics could be reinvented. The other one is the use of informal strategies of students, once they can be interpreted as preceding more formal ones. Developmental research conducted by Ter Heege (1983) and Streefland (1988) highlighted the suitability of students' solution procedures to lining out the course of instruction. According to them, student's alternative solution procedures are well suited for this purpose, acting as 'road signs' for the developer.

In general, a developer must find contextual problems that allow a variety of solution procedures, preferably those that, together, indicate a possible learning route through a process of progressive mathematising.

The reinvention principle implies long-term learning processes. These ones are not separated in learning steps that can be mastered independently, but instead they are processes of gradual changes. The intermediate steps are thus not to be seen as a goal in itself but it must be seen in a long-term perspective.

Lets elaborate a bit more about mathematising and the reinvention principle by following Gravemeijer (1994, chapter 3) in his discussion of the difference in the way applications are used in realistic mathematics instruction and in an information processing approach. In an information processing approach, mathematics is viewed as a formal system where its applicability is provided by general concepts and procedures, implying that one has to adapt this abstract knowledge in order to solve problems set in reality. The model that describes this process of solving a contextual problem with the help of formal mathematics, can be seen in figure one.

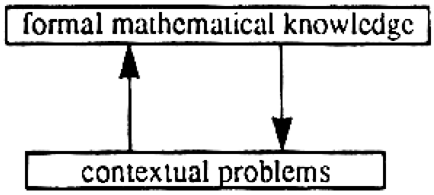

Figure 1. Application of formal mathematics (Gravemeijer, 1994, 92).

First of all, the problem has to be translated to a pre-designed system by formulating it in mathematical terms. After that, this problem is solved using mathematical tools and finally the mathematical solution is translated back into the original context. Transformation of a contextual problem into a mathematical one requires reduction of information in order to fit the pre-determined system and at the end of this process of problem solving, this information has to be taken into account. This translation process leads to recognising problem types and establishing standard routines, which contributes to a lack of insight in problem-solving. When a context problem is not recognised as a sort of problem that fits a standard pattern, students don't know how to handle it. Learning mathematics is then resumed to learning standard procedures to solve standard types of problems. It may be noted that in practice this process will not always be as linear as described, the problem solver may shuttle back and forth between activities of describing, solving and translating back.

In contrast, if one chooses to teach mathematics as an activity, problem-solving takes a different meaning. Teaching becomes problem-centred, which means that the problem is the actual aim of education, instead of the mathematical tools. The three usual stages of problem solving, namely describing the contextual problem to a more formal one, solving it in a more or less formal level and then translating its solution back to the context, also take place here, but in a different way. Formal mathematical knowledge is not a (pre-determined) system on its own where the problem has to fit in. Instead, one tries to describe the contextual problem in a way that allows us to understand and solve it. In fact, there's no talk about translation but of description.

The problem is described through schematising using self-invented symbols and through identifying the central relations in the problem situation. The problem doesn't have to be presented in commonly accepted mathematical knowledge. This description of the problem simplifies it, so that solving it in this more or less formal level differs a lot from applying a standard procedure. The final step of translating the solution back to the context, happens formally the same way as described before with the important difference that now symbols are meaningful for the problem solver as he was the one who gave them meaning.

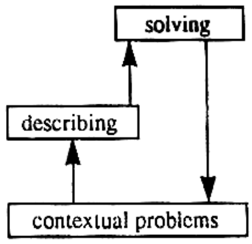

Figure 2. Realistic problem-solving (Gravemeijer, 1994, 93).

Another process takes place when students sequentially solve similar problems. Problem description and solving procedures develop, through simplifying and formalising, into an informal language, which in turn evolves into a more formal one, the commonly accepted mathematical language. This long time process is referred by Treffers (1987) as *vertical mathematisation*. Through this process of mathematising mathematical matter, formal mathematical knowledge is (re)constructed. This is distinguished from *Horizontal mathematisation*, which is mathematising contextual problems. Freudenthal characterises this distinction as follows:

> Horizontal Mathematization leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, compreendingly, reflectingly: this is vertical mathematization. The world of life is what is experienced as reality (in the sense I used the word before), as is symbol world with regard to abstraction. To be sure the frontiers of these worlds are vaguely marked. The worlds can expand and shrink-also at one another's expense (Freudenthal, 1991, p. 41-42).

As Freudenthal refers, the boundaries between what is vertical and what is horizontal mathematization has to do with what one understands as reality. Freudenthal (1991) clarifies: 'I prefer to apply the term 'reality' to that which at a certain stage common sense experiences as real.' (p.17) Reality and what one counts as common sense in not static but it grows through the learning experience of the person in question. In this context of reality as a mixture of interpretations and of sensual experience, mathematics can also become part of one's reality. In this sense learning mathematics is including it in one's reality. Freudenthal talks about 'Mathematics starting at, and staying within reality' (1991, p.18). Suitable contextual problems support this process, facilitating certain interpretations and strategies leading to horizontal mathematising processes.

Didactical phenomenology. This principle refers to the analysis of real-life sources of mathematics, in other words, the investigation of phenomena which (in the past) contributed for the development of a certain mathematical concept.

This has to do with the realistic mathematics idea that formal mathematics should develop from student's activity of generalising and formalising situation-specific problem-solving procedures and concepts. Therefore, this search for suitable contexts is necessary, firstly to find out the kind of applications that must come first in instruction (contexts which allow informal situation specific approaches) and secondly to study their suitability to give rise and stimulate the referred process of progressive mathematisation.

Emergent models. We shall come back to the function and character of models in realistic mathematics education and talk deeper about it. For now we shall keep it short.

This third Heuristic has to do with the role of emergent models as a means of support to help the students in building upon their informal knowledge to develop formal mathematics.

In product-oriented mathematics education, manipulatives are presented as pre-existing material models that students have to learn understand and master. In realistic education, instead, models emerge from students' activity itself as a model *of* a situation that is familiar to them. Through a process of generalising and formalising, the model becomes later on an entity on its own. Only then, can the model become a model *for* mathematical reasoning. Important in this process of character change of the model is that models initially derive their meaning from their reference to contextual situations.

Model is here used in a broad sense; it may refer to a verbal description, a model situation, a model procedure as well as ways of symbolising and notating. What characterises a model in RME is his function of simplifying, clarifying and summarising contextual or mathematical matter. At first models are thus context-specific, or models *of* concrete situations that are experientially real for the students. On this level, the model should allow solution strategies at the level of the situation where the contextual problem is defined. As the students solve similar problems, the model starts to become an entity of its own, which means that it can become an affordance for more formal mathematical reasoning.

According to this heuristic, we can sum up the criteria that emergent models have to fulfil:

- The model must not only refer to experientially real situations but it should also allow students to use informal solution strategies.
- The model has to encourage the process of progressive mathematising.
- The model must have the potential to become an entity in and of its own in the course of the learning process.

Use of models in Mathematics Education

The use of models in mathematics education has played an important place in instruction theories. Models take different forms and uses within different theories. In more traditional mathematics education theories the label 'model' refers to tactile material or diagrams that students can manipulate, and for this reason they are commonly referred in mathematics education as manipulatives. The way this manipulatives should be used varies from theory to theory.

We will discuss the problems of the use of manipulatives in instruction and analyse an alternative to its use. This alternative is given by a domain specific theory for realistic mathematics education, where the term model has a different character. Within this theory the label 'model' refers to a broader concept, which includes situation models, schemes, descriptions or ways of noting. Moreover, they are referred as "emergent", as they emerge from students' mathematical activity when solving mathematical problems in context.

Manipulatives in Mathematics education

In action psychology we have talked about Gal'perin's theory and his stepwise formation of well-formed mental actions. In his instruction plan, the use of manipulatives comes in the second step, right after the orientation phase. In his theory, the manipulative action is not necessarily made with material but symbolic representation can also be used (also referred as materialised action). He claims to be essential that manipulative action is isomorphous with the intended mental activity, so that this one can be well formed.

There are enough examples that show that working with manipulatives does not automatically fulfil this requirement. For instance, studies have shown that the use of the abacus as a manipulative material might get the children used to action structures, which do not correspond with the mental actions they are to conduct when doing written arithmetic (Gravemeijer, 1994, chapter 2).

But what does this isomorphism between the material and the mental action exactly mean? With Gal'perin's approach it seems that the students keep thinking about concrete material through out the instruction sequence. However, Parreren (1981) shows that students can ultimately leave any reference to the material manipulated. He shows with his " building block model" that actions at the beginning and end of the instruction don't have to be isomorphic. Instead, a number of actions at the beginning can be substituted by another later on, what he calls *shortcuts*. He distinguishes three of them: the forming of perceptive actions, the automation of motor skills and the one we are interested in, the restructuring of a task. This last means that at a certain moment of instruction, the student discovers he can substitute one or more actions (for instance, counting on) with another one (for example use of a property or a memorised fact).

With Van Parreren, we get a clearer idea of the mental action that is ultimately formed using manipulatives. However, he doesn't explain how students are set free from thinking about concrete materials. Let's see how information processing, or also called cognitive psychology, uses manipulatives during mathematics instruction.

Manipulatives in information processing.

As the name of the theory suggests, acquisition of knowledge is described as information processing where 2 types of phenomenon are distinguished. We have seen that knowledge is believed to be organised in a cognitive structure. Fitting a new knowledge element into this structure (expanding knowledge) is referred to as *assimilation*. A second phenomenon is called *accommodation*, and it means that sometimes the cognitive structure has to be completely reorganised in order to make room for new knowledge.

Within this theory, there is an important movement involved in an advance form of task analyses, the "Task analytical approach". In order to find a direction for instruction, the cognitive structures of beginners and experts are analysed. This approach, as it is known from Gagné (1977), was stripped of its behavioural features. However, the use of a top-down strategy is maintained. As the desired action, the action of the expert, is the starting point of this analyses. This places the to be learned expert procedures so much central that the aim towards acting with insight suffers the consequences.

> It seems as if the only question that is being asked is, how to get students so far that they will exhibit the discovered model behaviour, without asking oneself if the students understand what they are doing (Gravemeijer, 1994, chapter 2).

An example of instruction within this task analytical approach is the "mapping instruction". Children are asked to do subtraction problems using blocks (the so-called Dienes blocks) and using a written algorithm, maintaining a step-by-step correspondence between the blocks and the written symbols through the problem (Resnick & Omanson, 1987). It is claimed that, this way, calculation is made concrete as the mathematical relationships are embedded in manipulative material. However, blocks must be handled according to rules set by the researcher. For instance, the small blocks stand for the units, the bars for the tens and the squares for the hundreds. The problem with this approach is, according to Gravemeijer (1994, chapter 2), that the blocks are both the objects to be counted as well as the representation of the result of that count. This means that no distinction is made between the countable objects and the representation of the number, which makes children confused (see example in Gravemeijer, 1994, chapter 2, p. 62).

Resnick and Omanson (1987) concluded based on data analyses that a learning process of some other order is needed but the idea of working with this blocks remains. They believe that the connection between working with the blocks and doing arithmetic play an important role in helping children developing abstraction.

Criticism about the task-analytical approach. Cobb (1987) criticises the task-analytic approach on the formation of abstract mathematical objects. According to him, the analogy of working with the blocks and executing the written algorithms is only clear to the designer, as he was the one who created the units of ten or hundred as mathematical objects. As the student does not yet have that mathematical knowledge, he does not see this analogy. For instance, according to the task analytic approach, it is presumed that students recognise the bars as "tens", but studies have shown that this is not that simple. According to Cobb, this has to do with the fact that the concept ten is not an easy one for children. He refers that Steffe and Von Glasersfeld attested this as they identified six levels of the construction of "ten" as a mathematical object.

In order to judge the significance of learning material, Cobb refers to the importance of the distinction between the "actor's point of view" (the one of the student) and the "observer's point of view" (the one of the observer). One should try to look at the manipulatives through the eyes of the children, because the fact that we already have that abstract mathematical knowledge unable us to see the problems that children who not yet have it, go through. The problem has to do with the lack of distinction between the mental representation (internal representation) and a didactical representation in the form of concrete material (external representation), like Cobb points out. About this, Gravemeijer writes:

> By not making a clear distinction between internal and external representation it goes unnoticed that one is mixing up the time order: the student needs the mental representation which he or she must construe in order to be able to interpret the concrete representation (1994, chapter 2).

From a constructivist perspective, there is no fundamental difference between the so-called abstract mathematical knowledge and the situated informal knowledge of the students, since all knowledge is individually (re)constructed. Instead of trying to transmit abstract mathematical knowledge, Cobb suggests to provide students with the opportunity to construct their own mathematical knowledge themselves. Furthermore, through social interaction and negotiation, one can try to attune the various constructions within a classroom community as much as possible, and together achieve mathematical knowledge.

Alternative ways to avoid misconceptions. A way to destroy misconceptions of students is suggested by Van den Brink (1981). He suggests creating conflict situation in order to make students discuss their conceptions. These are situations that conflict with student's self-developed knowledge and make discussions possible and meaningful. Through discussions is then possible to realise these misconceptions and consequently destroy them.

However, one can instead wonder if it is not possible to avoid misconceptions on the first place. In RME, misconceptions are tried to be avoided through the use of the reinvention principle (see key principles of RME). In realistic mathematics, students attribute meaning to material devices by themselves in problem oriented instruction. For instance, to develop the denary system, concrete material is used as a visual support to symbolise quantities situated in context. Agreement about how to manipulate the material comes to the forth as a way to solve a problem and not as a pre-determined thing.

According to Gravemeijer (1994, chapter 2), this approach also brings up some problems. It is possible that children, at a certain point, start handling the material without thinking about the context or the meaning of the handling. Furthermore, as we have seen, students develop their own strategies that are not always isomorphous with the intended mental action. For this reason, it is important that action with concrete material is kept used in instruction as a transition phase. This means that they only should work as a framework or reference to other instruction actions. In other words, instruction must not be centred in the material action itself but in their support in solving a problem. Moreover, the material action should not be mechanised, in order to avoid students to keep on thinking about concrete material.

The realistic approach to manipulatives seems to offer solutions to avoid misconceptions, but according to Gravemeijer (1994, chapter 2), a study of the actual solution process of students and actual mental representation of mathematical concepts and relationships remains essential.

Models in RME

In RME a different approach is used to the use of models in instruction. Models are not presented as ready made things, mostly in the form of concrete material, that students have to learn and master in order to understand the abstract mathematical knowledge embedded in it. Instead, the label model refers to a broader concept, placed in an intermediary level between situated and formal knowledge. A model is defined in terms of signifying relations established in activity for some purpose (Cobb, 1998). Furthermore, models are not a result of extracting relationships from situations, but they rise from activity in and reasoning about situated problems. The firm distinction between the model and the situation being modelled disappears, as 'model and the modelled situation co-evolve and are mutually constitutive' (Cobb, 1998).

A characterising feature of models in RME is that they emerge in instruction, as a result of mathematical activity, following the reinvention principle and progressive mathematising. At first, they are context-specific models of a situation. Later on, as the model is generalised over situations, it changes from character, serving as basis for developing mathematical knowledge: it was a model *of* a context-specific situation to become a model *for* mathematical reasoning.

As we can see, the term model in RME is much broader than in manipulatives-based mathematics education. They include model situations, descriptions, ways of noting, and diagrams. They can also take the "external form" of a manipulative, but due to its place in instruction (not as a starting point) and to the way it is introduced (according to a bottom-up approach) makes it have another character. They are meaningful to the students as they are introduced in a more natural way, coming to the fore as the materialisation of student's own activity.

Conclusion

We have seen different approaches to the use of manipulatives in mathematics education. In each theory, manipulatives are used to facilitate the learning of abstract mathematical knowledge, through making it "concrete" to the students. Manipulatives are external representations of this knowledge. Instruction with manipulatives strives to maintain an isomorphism between the action with the material and the intended mental action.

The different approaches we have seen can basically be divided into two different kinds of approach.

One is a top-down approach, where manipulatives, obtained from abstracted mathematical knowledge, are designed by experts to create a concrete framework of reference in which the intended mathematical knowledge is embodied. Manipulatives are therefore presented to the students as an already-made thing that students have to understand and learn to use in order to learn the mathematical concepts. Knowledge is this way materialised and therefore made concrete to the students. Learning is then based on the idea of transfer, in other words, in the idea that the material transmits certain knowledge.

We pointed out that with this approach students don't get much insight and have problems with applications. In other words, they may succeed using manipulatives but they fail when they have to do it without it.

Analyses from a constructivist point of view have been able to explain this. According to Cobb the fact that the material is concrete in the sense of tangible does not yet mean concrete in the sense of making sense. The mathematical knowledge embodied in the material is only recognisable for the experts who already have it. For the students there is nothing to be seen, as they don't yet have that knowledge.

In contrast with this, we have seen a bottom-up approach, known as a domain specific theory for realistic mathematics education. Instruction starts at the informal knowledge of the student and models emerge out of student's mathematical activity. The term model refers to a concept broader than the concept of manipulative, including description of situations, ways of symbolising and making notations. Furthermore, in RME manipulatives loose their character of an already made thing. Materials are used as a tool to solve practical problems in a certain context, where meaning is negotiated between the members of the class.

Socio-constructivism and RME

Although realistic mathematics education and constructivism emerged independently, they have a lot of similarities. They are compatible and even complement each other (Gravemeijer, 1995 and Cobb, 1994).

In both constructivism and in RME, students' constructions are placed central, although in a slightly different way. In realistic education is said that students should construct their own knowledge while constructivism believes that there is no other way. Even if education doesn't take this into account, students always construct their own knowledge, say constructivists. Moreover, constructivism is more theoretical orientated than RME and in that sense it is possible to see constructivism as a theoretical basis for RME.

There are also other similar ideas between these two theories that are worth pointing out:

- The 'taken-as-shared' idea of constructivism is recognised in realistic mathematics as well, in Freudenthals's idea of common sense: 'reëel is datgene waar je niet over nadenkt, wat je zonder problemen als reëel of concreet aanvaardt'('real is all one doesn't think about, what one takes as concrete and real without problems') (Freudenthal, 1991). As well as in RME, socio-constructivists believe that general interpretations, knowledge and practices can be a point of departure in mathematics education.
- The "acculturation" idea of constructivism is in agreement with realistic believes, since learning mathematics is in RME seen as learning *to do* mathematics, to be able to participate in "mathematics as an human activity".
- The socio-constructivism idea that mathematics is a human construction ['de aard en de inhoud van de wiskunde is historisch, sociaal en cultureel bepaald' ('the nature and the content of Mathematics is historically, socially and culturally determined') (Gravemeijer, 1992)] is related to the idea of Freudenthal of mathematics "as an human activity".

In spite of all the similarities between these two theories, we can't forget that (according to Cobb, 1994) socio-constructivism is not an education theory like RME and, therefore, it doesn't define a particular way of teaching. As an epistemology (theory of knowledge) that it is, it describes knowledge development, whether teaching is going on or not.

Developmental Research

A big change in mathematics education has been happening in the last decades. In part influenced by Freudenthal's plea for a different kind of mathematics education a need for a new kind of mathematics was felt in the Netherlands. Internationally, the awareness for such a need coincided with the rise of constructivism.

In fact, although constructivism is not an instruction theory, it does give a good framework to think about mathematics learning in the classroom. Indeed, different constructivist studies demonstrated, for instance, the insufficiency of the traditional (Americans) primary mathematics education. Students' constructions were shown most of the times to be different from the ones intended, which brought researchers to think about the kind of mathematics learned through traditional methods.

In general, a change that gives more autonomy to the students is made in mathematics. Reform in mathematics education asks for a shift from what *teachers* do, to what *students* do, which means that "teaching by telling" has to be replaced by "students constructing" or "inventing", as Gravemeijer (1998a) notes. According to him, the problem that rises from this shift is how can one make students invent what one wants them to invent?

This is the question that researchers at Freudenthal Institute have been trying to answer in the last two decades, striving to develop a mathematics education, that correspond with Freudenthal's idea of "mathematics as a human activity". What they try to find out is the way mathematics education should look like so that it fulfils Freudenthal's educational philosophy. Having this as starting point, research is done through experimenting with mathematics education in practice, and reflecting on this practice. From this reflection, an empirically founded local instruction theory is developed for that topic, which in turn serves as base for new experiments. The result of this cyclic process of theory development is what we call a "domain-specific instruction theory for realistic mathematics education" (RME). The type of research that gives rise to this theory is the *developmental research*.

Developmental Research-developing RME

Developmental Research is a kind of research named by the Research Advisory Committee of the NCTM (1988) "Transformational research" (Gravemeijer, 1998a). This kind of research is characterised by its focus on "What ought to be" in place of "what is", once it addresses the question of how to create education that fulfils pre-given standards or ideals.

Developmental research is an integration of development and research, which

Gravemeijer very clearly describes as both "developing by research, and research by developing". He says:

> Developmental research consists of a mixture of curriculum development and educational research, in which the development of instructional activities is used as a means to elaborate and test as instructional theory (Gravemeijer, 1998a).

Having as point of departure Freudenthal's plea of "mathematics as a human activity", and his philosophical education theory, researchers experiment with mathematics education and reflect on this practice. An empirically grounded instruction theory develops from this reflection and constitutes the new base for further experiments. This theory initially has a local character as it describes the way a specific mathematical topic should be taught in order to fit the realistic philosophy. The step towards a more general theory is made when a pool of local instruction theories is formed and a search for common features is made. However, this theory is still local as it is confined to mathematics education, and it is linked to the realistic approach. The product of developmental research is for this reason called a "*domain specific instruction theory for realistic mathematics education*" (RME).

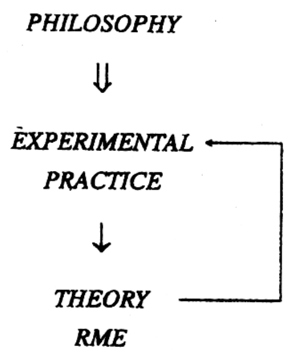

Figure 3. Cyclic process of theory development (Gravemeijer, 1998).

This way, curriculum development goes together with theory development as two sides of the same coin, having as final goal theory development.

The research method

The research method itself is also subjected to a similar cyclic process as the instruction theory, RME. See picture 4.

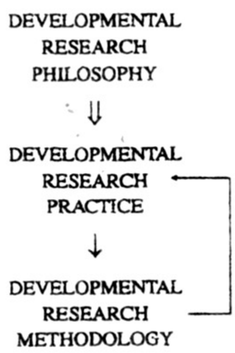

Figure 4. Cyclic process of emergence of the research methodology (Gravemeijer, 1998).

While developmental research practice shapes its method, the method itself guides the practice. This has to do with the research characteristic, of researching "what is ought to be" in place of "what is". Developmental research practice tests its method and provides information to improve it, becoming more effective in searching "what is ought to be". As we see, just like with the instruction theory of RME, the developmental research method is never ready but it is in permanent transformation and adaptation.

Instruction design

The approach to instruction design within developmental research differs a lot from traditional approaches. Traditional instruction design models are concentrated on learning outcomes without giving attention to the processes that lead to those learning results. On the contrary, in developmental research, these teaching-learning processes and students mental constructions are central.

The teacher's role is, like one can expect, other than in a traditional design mode. To understand this better, we introduce what Simon calls *mathematics teaching cycle*.

Simon's mathematics teaching cycle. An important notion in the teaching cycle of Simon, and which describes the role of the teacher, is the notion of *hypothetical learning theory* (HTL). He defines it:

> The consideration of the learning goal, the learning activities, and the thinking and learning in which the students might engage make up the hypothetical learning trajectory (…).

This concept refers to the teacher's work before the teaching situation. According to Simon, the teacher should try to envision the student's mental activities when working in the activities he plans, and furthermore try to anticipate how these student's mental activities might help them achieve the desired insights. He speaks about hypothetical learning trajectory, as the actual one can not be known in advance. What is possible is to construct a hypothetical one based on expectations about this learning trajectory. Of course, and according to constructivism philosophy, each student follows a slightly different individual learning trajectory, but they are quite similar.

During the teaching-learning process in the class, the teacher can find out in which extend hypothetical learning trajectories differ from the real ones. With new insights about student's conceptions and with the experience with the instructional activities, the teacher will be able to construct a modified hypothetical learning trajectory for the subsequent lessons, and so on. This is what Simon refers to as *mathematics teaching cycle*.

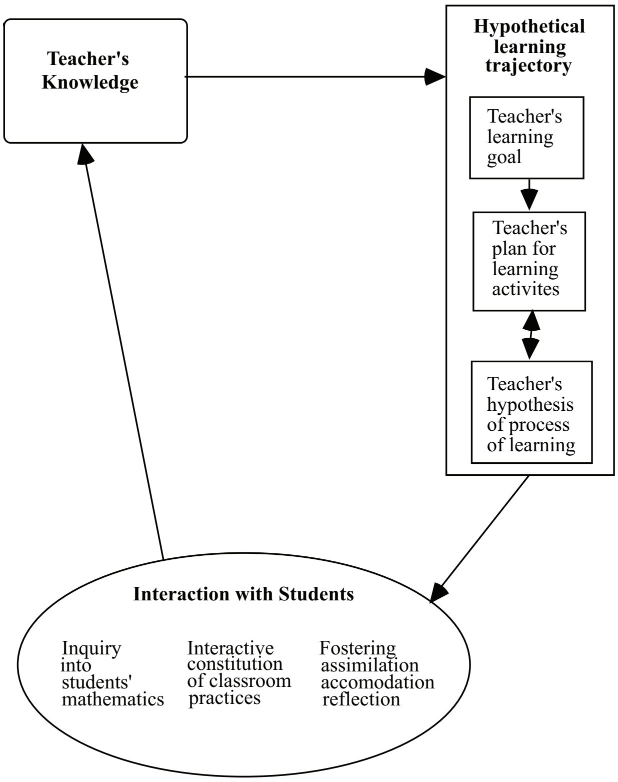

Figure 5. Mathematics teaching cycle (Simon, 1995, 136).

Freudenthal's cycling process. In a similar way Freudenthal talks about a cyclic process of "thought experiments" and "teaching experiments". See figure 6:

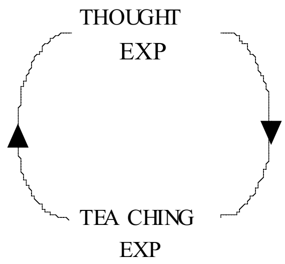

Figure 6. Cyclic process (Gravemeijer, 1998).

This cyclic process is the backbone of developmental research. This process is very similar to Simon's mathematical teaching cycles. The researcher constructs a set of instructional activities, which is worked out in a process of (re)designing and testing. There are however some differences. First of all, the researcher's goal is not only to develop instructional sequences but especially local instructional theories. Furthermore, while the teacher may focus on one or two lessons, the developmental researcher has to have a long-term learning process in mind. The process he goes through is then a chain of cycles of thought and teaching experiments. See figure 7:

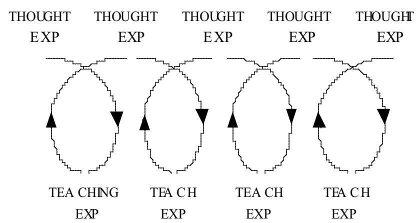

Figure 7. Developmental research, a cumulative cyclic process (Gravemeijer, 1998).

Justifications

Since the theory and conclusions that develop from developmental research have an empirical base, it brings up problems of justification. However, Freudenthal explains it very simply:

> Developmental research means: experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience (1991, p.161).

The process through which the local instruction theory is developed should justify the theory. This is called in ethnographic research “trackability”. As Gravemeijer refers (1998a), since the norm of trackability is truly fulfilled, the teachers can appropriate the experiences and considerations of the researcher and be able to make their own adaptations. They can take the theories of the researchers as conjectures, and test them in their own classes, making a contribution to the development of those theories, “instead of being passive consumers of knowledge produced by others”.

Conclusions

We have seen that nothing in developmental research is static or ready, neither its method nor the resulted domain-specific theory for realistic mathematics education. Developmental researchers are themselves under a cyclic learning process of thought and teaching experiences, through which they construct theory and improve their method. In this process, conjectured theories are constantly adapted and transformed to fit the new “reality” instructional practices reveal.

Furthermore, nothing is considered alone but everything is connected. So we talk about “developing by researching, and researching by developing” (Gravemeijer, 1998a).

Note that this idea that in developmental research, theory and methodology are constantly under construction is in agreement with constructivism and with realistic mathematics philosophy, where it has its base of inspiration and guide line. Mathematics is an human activity and learning is a process through which knowledge is constructed. We can’t say for sure if something is true or really exists. Something is true only while it is seen as so. Moreover, everybody is constantly constructing, and adapting knowledge based on his or her perception and experiences, thus also researchers.

Final remarks

After all that has been discussed, I would like to point out two remarkable things.

First of all, I would like to highlight the never-ready character of the domain specific theory for realistic mathematics education and its research method, the Developmental Research. Its character works as a protection to avoid that it becomes unadjusted to reality contributing for a more reliable theory. History has proved enough times that ‘trues’ change through out time. New discoveries ask for constant adaptations and rearrangements of what is taken as true at a certain moment in time. In other words, Human knowledge is under a constant process of reconstruction. The fact that the method of developmental research and the theory that arises from it are in constant adaptation and change, based on practical experience, suffering a development itself, makes it stay actual with current beliefs, mastering itself.

Another thing I would like to point out, as I find it of major importance in education, is that in realistic mathematics education, as well as in the socio-constructivism, the students are placed central. It is the student who constructs its own knowledge, say socio-constructivists. Students should have the opportunity to experience a learning process similar to the one followed by mathematicians, reinventing mathematics through guided reinvention and progressive mathematising, is the opinion of supporters of realistic mathematics education. Models should "emerge" from the students' own mathematical activity, working as a natural tool in problem solving, and not as something made by experts to transfer to students the mathematical knowledge embodied in it. More responsibility and autonomy is given to the student, making a clear contrast with the "God" authority of the teacher of more traditional instruction theories. In sum, a bottom-up approach is preferred to a top-down one, as being a more natural one.

I could say that I defended a top-down approach to instruction, as that was the one I experienced myself at school most of the times. However, I remember not being very enthusiast about it. I remember asking myself what the teacher was actually trying to say, and asking myself, sometimes also to the teacher, about the reason for those names and symbols, where they had come from or what their use was. I am also aware of the lack of insight I have in a lot of subjects of mathematics and that I have to use tricks and mechanised methods to solve some problems or to do some computations. Furthermore, I notice that I can't learn something that doesn't fit properly in the knowledge I have at a certain point and that, therefore, I construct my own knowledge having as starting point what I know at that moment. As a conclusion, I definitely defend a bottom-up approach to instruction, where mathematics "starts and stays within reality" and knowledge grows in a meaningful way.

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ABSTRACT. Realistic Mathematics Education (RME) has been developed in the Netherlands in the past decades, based on the educational philosophy of 'mathematics as an human activity' of Freudenthal. Starting with a brief history of theories on learning and instruction, and passing through a discussion on the use of models in Mathematics Education, in this article, a choice for the bottom-up approach to learning and instruction of Realistic Mathematics Education is gradually justified. In this process, the key principles of RME as well as the type of research that gives rise to it are discussed as well.

Key-words: *Realistic Mathematics; Teaching and learning theories; Developmental Research; Models in Mathematics Education; Manipulatives.*

RESUMO. A Educação Matemática Realista (RME) tem vindo a ser desenvolvida na Holanda nas últimas décadas, baseada na filosofia educacional da 'Matemática como uma actividade humana' de Freudenthal. Esta teoria de Educação Matemática faz uma abordagem ao ensino e aprendizagem que se distingue de outras, entre outras razões, pela sua abordagem "bottom-up". Começando por delinear um percurso histórico sobre as diferentes teorias de ensino e aprendizagem e passando por uma discussão acerca do uso de modelos em educação Matemática, justifica-se gradualmente neste artigo a importância desta abordagem. Neste processo, discutem-se ainda os princípios chave que guiam a RME e o tipo de investigação que a origina.

Palavras-chave: *Matemática realista; Teorias de ensino e aprendizagem; ´Developmental Research´; Modelos em Educação Matemática; Material Manipulativo.*