Integrating history of mathematics into the mathematics classroom[1]

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"*Learning is both cognitive and affective.*
So, too, are the mathematics problems of history"
(Swetz, 1989, p. 376).

Introduction

Does the history of mathematics have a role to play in mathematics education? If so, how can it be integrated into current mathematics curricula, and what are the new challenges teachers face when history of mathematics is an integral part of their teaching? These questions, which have concerned many mathematics educators for a long time, set the tone for this paper, which is not aimed at being an original text, but rather at sensitizing teachers to the importance of the history of mathematics for classroom teaching and at informing them about different possible ways of adequately integrating history of mathematics into instruction.

Although many references will be made to Portugal and the United States of America (the native countries of the authors), this paper is neither about those countries nor about the role played by the history of mathematics in their educational systems[2]. Nonetheless, a certain focus will be put on the Portuguese situation concerning the integration of history of mathematics into its school curricula, mainly those pertaining to the more recent reform documents and proposals for grades 7 through 12 (e.g., Abrantes, Serrazina, & Oliveira, 1999; APM, 1988; DGEBS, 1991; DES, 1997, 2002).

We will begin this paper by elaborating on the reasons the history of mathematics should be integrated into mathematics instruction, indicating a number of benefits for both teachers and students resulting from that integration. Some objections and barriers to the integration of history of mathematics into classroom teaching will also be put forward. The second part of this paper is aimed at informing teachers about a number of ways of integrating history of mathematics into their classroom teaching, providing them with several useful resources and references. It will become clear throughout the text that we believe an appropriate integration of history of mathematics into instruction has much to contribute to the improvement of the teaching and learning of mathematics.

Part I

Should the History of Mathematics Be Integrated into Classroom Teaching?

There are many different answers to this question, although the vast majority of them is favorable. Indeed, "the careful and judicious use of the history of mathematics" (Rickey, 1996, p. 252) can become a tool for effective teaching (e.g., Wilson & Chauvot, 2000), and it can help both teachers and students meet the challenges posed by the current reform recommendations for mathematics education (e.g., Abrantes et al., 1999; APM, 1988; DGEBS, 1991; DES, 1997, 2002; NCTM, 1991, 1994, 1995, 2000). In particular, it can stimulate and develop students' mathematical communication skills and understanding of mathematical connections, and it can foster students' appreciation for mathematics (e.g., Arcavi, Bruckheimer, & Ben-Zvi, 1982; Bidwell, 1993; Fauvel, 1991; Tzanakis & Arcavi, 2000; Wilson & Chauvot, 2000).

"There has been interest over several centuries in the relations between the history of mathematics and the teaching and learning of mathematics" (Fasanelli, 2000, p. 33). For example, J. Bernoulli assigned L. Euler some classical historical problems during his university studies as early as in 1725 (Calinger, 1996), and the *Estatutos da Universidade Portuguesa* (Statutes of the Portuguese University), written under the scope of the Reforms of the Portuguese Universities in 1772, recommended that teachers and students use and associate the teaching and learning of mathematics with its history (Estrada, 1993). A century later, K.

Weierstrass stressed the importance of carefully and simultaneously studying the classics and research journals as being essential for the understanding of the foundations of mathematics and for future original contributions to the universal body of mathematical knowledge (Calinger, 1999). By the middle of the 20th century, the Portuguese mathematician, Prof. J. Sebastião e Silva, also emphasized the importance of knowing historical developments (with special attention to the history of mathematics in Portugal) for better understanding current mathematical concepts and procedures (Silva, 1993, 1995). More recently, several meetings and conferences (gathering historians, mathematicians, mathematics educators, researchers, and national and international mathematical and educational organizations) have taken place throughout the world, including the Quadrennial Meeting of the International Group on the Relations Between History and Pedagogy of Mathematics, held in Portugal, in 1996, and the first three European Summer Universities on History and Epistemology in Mathematics Education, held in France, Portugal, and Belgium, in 1993, 1996, and 1999, respectively.

In Portugal, there are some organizations that have played a significant role in sensitizing and informing teachers about the history of mathematics and about its educational value. For example, the *Grupo de Trabalho sobre História e Ensino da Matemática* (GTHEM) [Working Group on the History and Teaching of Mathematics], which is affiliated with the *Associação dos Professores de Matemática* (APM) [Association of Teachers of Mathematics], was founded in 1993 due to the increasing interest on the integration of history of mathematics into instruction on the part of the APM members and classroom teachers in general. The GTHEM has been providing classroom teachers with useful information, documentation, and publications about the history of mathematics and its use in the classroom (e.g., GTHEM, 1997, 2000), including biographies of mathematicians, historical overviews of specific mathematical themes or certain time periods, some activities for classroom use, and reports of a number of innovative experiences and investigations about the integration of history of mathematics into teaching (APM, 2002).

The *Sociedade Portuguesa de Matemática* (SPM) [Portuguese Society of Mathematics], founded in the 1940's, has been periodically organizing the *Seminário Nacional de História da Matemática* (SNHM) [National Seminar on the History of Mathematics]. In pursuing its main goal of promoting interest for the history of mathematics (although not necessarily related to its educational use), the SPM has organized frequent talks and colloquia at the school level and several national and international meetings with special emphasis on the history of mathematics in Portugal (Saraiva, 1993).

In spite of an increasing global attention towards the integration of history of mathematics into school mathematics, classroom realities have been somewhat discouraging (e.g., APM, 1998; Fasanelli, 2000; Fauvel, 1991; Nunes & Guimarães, 1994). Teachers have been told *about* the benefits of integrating history of mathematics into their classroom teaching, but they have been "given little guidance on *how* to do so" (Fasanelli, 2000 p. 4; Rogers, 1991). In general, teachers have little knowledge and preparation from their teacher education programs about integrating the history of mathematics into their teaching. Therefore, there has been a worldwide increased concern about educating both pre- and in-service mathematics teachers about history of mathematics and its integration into the classroom (e.g., Silva & Araújo, 2001; van Maanen, 1997). However, teacher education programs all over the world usually do not offer courses on history of mathematics or, if they do, they tend to be optional and often completely neglect the didactical value of history of mathematics (Fasanelli, 2000).

Portugal is somewhat an exception since most universities offer a history of mathematics course—mainly in secondary teacher education programs—whose syllabi are not necessarily similar. For example, as far as the information provided in Fauvel and van Maanen's book (2000) is concerned, at the University of Coimbra, much attention is given to Portuguese history, specifically to the period of the voyages of discovery (15th and 16th centuries), and to the achievements of Portuguese mathematicians since then. On the contrary, at the University of Porto, the emphasis is put into the contributions of the Greeks, with a few references to mathematics in ancient Babylonia and Egypt, leaving future mathematical developments for Master's programs (Schubring, 2000). In any case, the didactical use of history of mathematics in school mathematics, which is desirable in a mathematics teacher education program, is not emphasized. Although

> the history of mathematics is not a panacea for all the problems of preparing mathematics teachers, … [it] is a good vehicle for reflecting on cognitive and educational problems, for working on students' conceptions of mathematics and its teaching, and for promoting flexibility and open-mindedness in mathematics (Furinghetti, 2000, p. 51),

and this holds for students too (Barbin, 2000). Indeed, integrating the history of mathematics into classroom instruction will not result in miraculous changes in student motivation to learn or in student achievement (Fauvel, 1991). However, it can provide a new perspective about mathematics as a human endeavor. We will come to this later.

Lack of knowledge about history of mathematics and how it can be integrated into the mathematics classroom is not the only factor accountable for the state of affairs in today's classrooms across the world. Teachers' beliefs about the nature of mathematics and about its teaching and learning deeply influence their classroom practices (e.g., Ernest, 1989; Swafford, 1995; Thompson, 1992) and consequently their willingness to integrate history of mathematics into their teaching. Indeed, if mathematics is seen as a fixed and finished body of knowledge, and if teaching mathematics is seen as the transmission of knowledge from teachers to students, then there is hardly room for the history of mathematics in the teaching and learning processes. On the contrary,

> if mathematics is seen as one of many forms of knowledge, or even as a kind of cultural manifestation or human activity, then the history of that subject will be meaningful and the study of that history will become a means of better understanding the relationships between mankind and the mathematical knowledge, within a certain cultural context. (Silva & Araújo, 2001, p. 19-20)

Still another reason why history of mathematics has not been integrated into school mathematics concerns the ways, if any, in which textbooks typically address the history of mathematics, mostly limited to an *inclusion* of a few historical notes (generally, biographies and curiosities) at the end of each chapter (e.g., Silva, 1993; Fasanelli, 2000). This fact leads teachers to see history of mathematics as separated from the curriculum and as an "alien to everyday classroom work" (Jahnke, Knoche, & Otte, 1996, p. viii). Therefore, teachers typically ignore the often shallow recommendations about integrating the history of mathematics into the official curriculum (Fasanelli, 2000).

Concerning the integration of history of mathematics into classroom instruction, the recommendations of the Portuguese official mathematics curricula, not only those currently in use but also those which are expected to be implemented shortly, continue to be fairly vague, almost restricted to methodological suggestions (Abrantes et al., 1999; Silva, 1993; DGEBS, 1991; DES, 1997, 2002), as it is the case in most countries. In other nations, such as Italy, Denmark, and the United States of America, "the association of history with mathematics teaching has a long tradition" (Fasanelli, 2000, p. 9), and many publications have been released to provide both teachers and students with an overview of the history of mathematics and with resources for classroom use (e.g., Calinger, 1996, 1999; Fauvel, 1990a, 1990b; Fauvel & van Maanen, 2000; Jahnke et al., 1996; Katz, 1993, 2000; Laubenbacher & Pengelley, 1999; NCTM, 1969; Selin, 2000; Struik, 1986; Swetz, Fauvel, Bekken, Johansson, & Katz, 1995)[3].

In many countries (Portugal included), along with the little integration of the history of mathematics into classroom teaching, there has been limited research on the topic (e.g., Furinghetti, 1997; Silva, 1993; Stander, 1989). Nonetheless, a number of research studies have indicated many benefits (which we will discuss in the next section) from integrating history of mathematics into mathematics education, for teachers and students at all grade levels (e.g., Barbin, 2000; Furinghetti; Isaacs, Ram, & Richards, 2000; Marshall, 2000; van Ameron, 2001).

Before proceeding, however, a word is necessary about our preference of using the expression *integrating* as opposed to *using* history of mathematics into mathematics teaching, a preference that is shared by many (e.g., Estrada, 1993; Furinghetti, 1997; Wilson & Chauvot, 2000). Merely *using* history of mathematics in classroom teaching typically leads students to feel it is just another finished piece of mathematical knowledge, something that happened a long time ago and has no actual usefulness for nor meaning to them, other than possibly being interesting (e.g., Barbin, 1991; Estrada; Furinghetti & Somaglia, 1998). *Using*, as opposed to *integrating*, the history of mathematics for motivating student to learn, for example, can lead to a rather negative effect. Indeed, if teachers simply present historical aspects to their students, the most likely effect is losing students' interest in the subject (Thomaidis, 1991), "as the student[s] very quickly learn that the real stuff comes only after the 'storytelling' is over" (Otte & Seeger, 1994, p. 352). On the contrary, truly *integrating* history of mathematics into the teaching of this subject leads students to view history of mathematics as the linking agent that ties mathematics together through time and across its many areas, to experience mathematics in the making, and to see mathematics as a dynamic and creative human enterprise (e.g., Barbin; Grugnetti & Rogers, 2000; Reimer & Reimer, 1995; Tzanakis & Arcavi, 2000). This last approach to history of mathematics in the classroom is the one that resonates with the currently dominant perspectives on mathematics education in Portugal (e.g., Abrantes et al., 1999; APM, 1988; DGEBS, 1991; DES, 1997, 2002; NCTM, 1994).

Benefits from Integrating History of Mathematics into Teaching

Motivation. Integrating history of mathematics into classroom teaching has been found to have many benefits for both teachers and students. For example, it is a means of motivating students to learn (e.g., Ernest, 1998; Fauvel, 1991; Reimer & Reimer, 1995; Tzanakis & Arcavi, 2000). The history of mathematics is full of emotional instances whose integration into classroom teaching is capable of capturing students' attention and curiosity about mathematics (e.g., Barbin, 2000; Rubinstein & Schwartz, 2000; Swetz, 1984), such as the *drama* in Galileo's challenge of the Ptolemaic views of the motion of planets, the *mystery* in Egyptian numerology and in the existence of mathematical constants as π and e, the *adventure* in the search for a ruler and compass construction for increasingly ordered regular polygons, and the *intrigue* in the analyses of certain "secrete numerical codes that were popular in the Middle Ages" (Swetz, p. 55). Furthermore, teachers may also feel motivated to deepen their knowledge about mathematics, although, entering the world of history of mathematics and seeking to integrate it into classroom instruction may constitute a challenge (e.g., Estrada, 1993) many teachers might not be willing to pursue.

Instances such as the ones mentioned above may intrinsically motivate students to want to learn more. In fact, intrinsic motivation to learn is preferable to an extrinsic one (e.g., Middleton, 1995) as the former is characterized by students' desire "to engage in learning for 'its own sake'" (Middleton & Spanias, 1999, p. 66), and students' interest for classroom activities, which they also find enjoyable, challenging, and useful for life (Middleton & Spanias, 1999; Schiefele & Csikszentmihalyi, 1995). On the contrary, extrinsic motivation is characterized by students' wish to engage in academic tasks to obtain rewards or avoid punishment, that is, they are focused on "obtaining favorable judgments of their competence from teachers, parents, and peers," (Middleton & Spanias, 1999 p. 66) and on "showing evidence of ability … by outperforming others, or by achieving success with very little effort" (Ames & Archer, 1988, p. 260).

Teachers' knowledge, beliefs, and classroom practices. The integration of history of mathematics into classroom instruction is an exciting challenge to teachers' knowledge and creativity (Estrada, 1993). Besides better understanding of mathematical ideas, teachers tend to "to change the way they think about their students" (Barbin, 2000, p. 64), thus seeing them as thinking and inquiring beings. Therefore, teachers' beliefs tend to increasingly resonate with the philosophical perspectives and educational goals that dominate in the Western countries, including Portugal and the United States of America (Abrantes et al., 1999; APM, 1988; Barbin; DGEBS, 1991; DES, 1997, 2002; NCTM, 2000). Consequently, teachers' classroom practices also tend to improve, ultimately affecting student learning in a positive way, although a change in beliefs is not necessarily translated into a change in classroom practice (e.g., Franke, Fennema, & Carpenter, 1997; Raymond, 1997; Thompson, 1992).

Attitudes, anxiety, and creativity. The *integration* of history of mathematics into school mathematics has been shown to improve students' and teachers' perceptions of the subject, as well as their attitudes, creativity, and enthusiasm about it (e.g., Arcavi et al., 1982; Barbin, 2000; Ernest, 1998; Estrada, 1993; Fauvel, 1991; Marshall, 2000; Philippou & Christou, 1998; Rubenstein & Schwartz, 2000; van Maanen, 1997). Furthermore, students' usual anxieties about mathematics may also be diminished. Indeed, traditional teaching leads to the compartmentalization of mathematics, which "produces students who perceive mathematics as an incomprehensible collection of rules and formulas that appear en mass and threateningly descend on them" (Swetz, 1984, p. 54). Consequently, "students build psychological barriers to true mathematical understanding and develop anxieties about the learning and use of mathematics" (p. 54). However, when historical aspects are integrated into the classroom instruction, students' mathematical anxiety and mathematical avoidance can be significantly reduced (e.g., Marshall, 2000; Schubring, 2000).

Learning difficulties. There is a connection "between subject matter that is difficult for a student to grasp and that which was difficult for the mathematical community to accept" (Kelley, 2000, p. 16). Thus, through the study of history of mathematics and its proper integration into mathematics instruction, teachers become more sensitive to and better able to understand and address the obstacles students encounter in learning several mathematical concepts and procedures (e.g., Fauvel, 1991; Tzanakis & Arcavi, 2000): for instance, the concept of zero, 0!, negative numbers and their operational rules, complex numbers, series, limits, and non--Euclidean geometry (e.g., Blake & Verhille, 1985; Ernest, 1998; Grugnetti, 2000; Kleiner, 1988). Since students, at various grade levels, find learning obstacles that are similar to those of past mathematicians (e.g., Radford, Katz, Dorier, Bekker, & Sierpinska, 2000), the history of mathematics provides classroom "teachers with a tool for anticipating psychological problems in the learning of mathematics" (Ernest, p. 26). Indeed, teachers' mathematical literacy is enriched as well as "their didactical repertoire of [questions, responses,] explanations, examples, and alternative approaches to present a subject or to solve problems" (Tzanakis & Arcavi, p. 206; see also Barbin, 2000; Tzanakis & Thomaidis, 2000). Through the integration of history of mathematics into classroom teaching, students themselves feel they are not alone when they struggle with so many problems during their learning of mathematics, and they see the subject less frightening (Ernest; Fauvel). Students' "confidence increases with the realization that the great [mathematicians] did not write down their answers immediately" (Stander, 1989, p. 241), and with the recognition that mistakes, doubts, intuitive arguments, "controversies, and alternative approaches to problems are not only legitimate but also an integral part of mathematics in the making" (Tzanakis & Arcavi, p. 205).

Engagement in work. Current views of mathematics education stress the importance of having students appreciate the value and worth of working difficult problems and challenging investigations, instead of giving up after having encountered the first obstacles (Abrantes et al., 1999; DES, 1997, 2002). In fact, the history of mathematics is full of examples of ups and downs throughout the developmental path of this body of knowledge (Swetz, 1984). Students may realize the "tedious trial-and-error process behind mathematical discoveries" (Horn, Zamierowski, &

Bayer, 2000, p. 688) and the time-dependent nature of such concepts as proof and rigor (Tzanakis & Arcavi, 2000). Most importantly, however, students may increasingly appreciate the value and "power of hard work and determination" (Reimer & Reimer, 1995, p. 105), as well as the importance of persistence in pursuing one's goals. The lives and work of Kepler and Andrew Wiles, for instance, can serve as examples "of hope, disappointment, persistence, and ultimate triumph" (Shotsberger, 2000, p. 680; Tzanakis & Arcavi).

Classroom discourse. Integrating history of mathematics into mathematics teaching can stimulate classroom discourse. For example, it can help teachers and students explain and answer many of the why questions that may arise in the classroom (e.g., Bidwell, 1993; Kelley, 2000). Such questions are likely to come from students who are not willing to accept what the teacher says without a certain amount of external support (Reimer & Reimer, 1995). For instance, students might ask questions about the origins of certain computational methods, notations, and words we currently use within the mathematical community (e.g., Estrada, 1993; Rubinstein & Schwartz, 2000; Tzanakis & Thomaidis, 2000). In fact, mathematical terms can be viewed as "preserved fossils from olden times, and digging them up can result in a fascinating discovery of how mathematics [has] evolved" (Rubinstein & Schwartz, p. 664).

Connections. The integration of history of mathematics into school mathematics is an excellent means of recognizing the many connections that exist among the various mathematical areas and between mathematics and other subjects (e.g., Furinghetti & Somaglia, 1998; Grugnetti & Rogers, 2000; Reimer & Reimer, 1995). For example, teachers can demonstrate the relationships among arithmetic, algebra, and geometry by drawing on the work of Euclid, Al-Khwarizmi, and Descartes, and strong links among mathematics, geography, engineering, and astronomy can be (interestingly) found in the development of navigation instruments and calculation devices, so crucial for the success of the (Portuguese) voyages of discovery (Grugnetti & Rogers, 2000; Veloso, 1994).

Math as a living science. Studying the historical development of mathematics allows both teachers and students to realize the evolutionary character of this science. They can see how mathematical ideas were generated in the past, and how many of the concepts they deal with in their classrooms and everyday lives, as well as several techniques they use today to solve problems, have been around for quite a long time (e.g., Estrada, 1993; Fauvel, 1991; Furinghetti & Somaglia, 1998; Swetz, 1984; Tzanakis & Arcavi, 2000). Current mathematical theories, ideas, and procedures can be seen as "polished diamonds that started off as rough pieces of carbon" (Davitt, 2000, p. 692). Indeed, the history of mathematics is a living science that “creates a bridge from the past to the future” (Reimer & Reimer, 1995, p. 107), and it shows “how the conjunction of the old and the new is a commonly occurring event in the development of mathematics today” (Grugnetti & Rogers, 2000, p. 55). For example, students can see that there are problems that take many centuries to be solved, as Fermat’s Last Theorem and the Four-Color Theorem, and that there are also many problems still unsolved today, such as the Goldbach’s conjecture. Furthermore, they can better appreciate how mathematical “notation, terminology, computational methods, modes of expression, and representations” (Tzanakis & Arcavi, p. 205), have evolved throughout so many centuries, and how language and means used to convey and communicate mathematical knowledge have changed through times (e.g., Grugnetti, 2000; Oliveira & Ponte, 1999; Thomaidis, 1991).

Historical “material can breathe life into mathematical lessons” (Swetz, 1984, p. 55), and, therefore, school mathematics can become substantially humanized by means of a proper integration of history of mathematics into mathematics teaching (e.g., Bidwell, 1993; Reimer & Reimer, 1995; Swetz). The history of mathematics provides a vast number of "examples of the role that innovation and creation have played in the development of the field" (Brizuela, 1997, p. 4), and students can realize that mathematics is, in fact, a creative and cultural human endeavor (Barbin, 1991; Reimer & Reimer; Tzanakis & Arcavi, 2000).

Math and different cultures. Classroom teaching characterized by a historically-integrated perspective allows students to recognize and acknowledge the cultural, political, social, and economical contexts of mathematical developments, as well as the important roles that have been played by several distinct cultures in the evolution of mathematics (e.g., Barbin, 1991; Bidwell, 1993; Grugnetti, 2000; Swetz, 1984; Thomaidis, 1991). According to D’Ambrosio (1996),

> ethnomathematics lies at the boarder between history of mathematics and cultural anthropology. It may be conceptualized as the study of techniques (*tics*) developed in different cultures (*ethno*) for explaining, understanding, and coping with (*mathema*) their physical and socio-cultural environments. (p. 245)

In particular, ethnomathematical studies encourage both teachers and students to try "to understand how mathematics continues to be culturally adapted and used by people around the planet and through time" (D’Ambrosio, 2001, p. 309), and the history of mathematics provides students with numerous opportunities for investigating ancient cultures and societies (e.g., Gorman, 1997; Kelley, 2000), and for knowing, respecting, and valuing different cultures, including their own, from past to present days (e.g., Fauvel, 1991; Grugnetti & Rogers, 2000; Reimer & Reimer, 1995; Zaslavsky, 1991).

Indeed, the integration of history of mathematics into school mathematics opens up the doors for a multicultural approach to teaching (Grugnetti & Rogers, 2000), in which multiculturalism must be seen as

> the identification and celebration of diversity, the respecting and valuing of the work of others, the recognition of different contexts, needs, and purposes, and the realization that each society makes and has made important contributions to the body of knowledge we call mathematics. (p. 51)

In addition, students may realize how societal norms and practices of several cultures have influenced mathematical developments (for example, the political and religious influences on the work of Galileo), and how mathematics has influenced "the ways that people operate in, and think about, the world" (Wilson & Chauvot, 2000, p. 643), such as the new information and communication systems that are so central to today's world life.

Objections and Difficulties to Integrating History of Mathematics into Teaching

Despite the many benefits an adequate integration of history of mathematics into classroom teaching can bring to both teachers and students, some arguments "challenging the desirability or feasibility of seeking to integrate history of mathematics in[to] mathematics education" (Tzanakis & Arcavi, 2000, p. 202) have been put forward. Among such arguments we find that: (1) history of mathematics is not mathematics; (2) history of mathematics may confuse the students rather than help them better understand the subject; (3) students' typical poor sense of the past may hinder them from historically contextualizing mathematical developments; (4) teachers lack clear directions on how to consistently incorporate historical components in their students' assessment and they are overly concerned about examination scores, a concern that is shared by parents and students; (5) integrating history of mathematics into classroom teaching is very time consuming; (6) there is a generalized lack of resources for classroom use; and (7) the vast majority of classroom teachers lack knowledge and expertise in history of mathematics and in how it can be integrated into mathematics instruction (Fauvel, 1991; Fowler, 1991; Tzanakis & Arcavi).

Though still quite scarce, research on the integration of history of mathematics into instruction has suggested that most of the arguments listed above have not found significant empirical support, except, perhaps, for the last two arguments (Fauvel & van Maanen, 2000). Indeed, a true integration (rather than a mere usage) of history of mathematics into mathematics lessons is able to overcome the five first arguments of the list. However, it is essential that teachers be better educated in the history of mathematics and in its integration into classroom teaching, and that there be adequate classroom resources for such an integration. Much research is needed to address the aforementioned issues, not only from an empirical perspective (which is certainly useful, particularly to classroom teachers) but also, in our opinion, from a theoretical one.

Integrating history of mathematics into classroom teaching is not an easy task (e.g., Fauvel, 1991). Teachers do lack knowledge about the history of mathematics and its educational value and potential (e.g., APM, 1998; Fasanelli, 2000; Thomaidis, 1991; Tzanakis & Arcavi, 2000), which might explain why they are typically reluctant to try an instructional approach that follows a historical perspective (Reimer & Reimer, 1995). Knowing history of mathematics is essential for its adequate integration into mathematics lessons (Bidwell; Katz, 1997; Tzanakis & Arcavi, 2000). In this respect, both teacher education and professional development programs have a crucial role to play, as they "should 'fill in the pieces', supply the 'why', 'where', and 'how' for the many concepts [teachers may have] already learned" (Swetz, 1995a, p. 99). In other words, teachers' knowledge of history of mathematics should itself be integrated with pedagogy (Freudenthal, 1981).

Still, within any teacher education or professional development program, no single course on the history of mathematics can cover all mathematical developments through thousands of years involving thousands of people. Therefore, history of mathematics instructors should carefully select instances of the history of mathematics that are relevant for both developing teachers' knowledge about it and about its integration into school mathematics. A discussion of a set of criteria for this selection is out of the scope of this paper, but we leave it open for a later discussion.

Although it is largely acknowledged that there is a widespread lack of resources about the history of mathematics and about its integration into classroom teaching, some information about it can be relatively easy to find, either in print or over the internet (e.g., Barrow-Green, 1998; Fauvel, 2000; Marshall & Rich, 2000). Besides some books referenced in the first part of this paper, the WWW is also full of information about the historical developments of mathematics. For example, we can find in the website *http://www.br.groups.yahoo.com/group/hist-mat-port* a discussion and information forum, in Portuguese, which we believe is of great help for Portuguese-speaking classroom teachers who seek to know more about the history of mathematics.

However, electronic information must be carefully assessed for quality and reliability by evaluating the authority of the author, the accuracy and currency of information, the links to other sites, and some other general characteristics as the quality of the text, the completeness of the information, and the ease of navigation (Barrow-Green, 1998).

On the other hand, information about how to integrate history of mathematics into the classroom is still scarce (Fasanelli, 2000; Rogers, 1991; Tzanakis & Arcavi, 2000), although some efforts have been done lately in order to address this issue. For instance, several professional journals have devoted entire issues to the integration of history of mathematics into classroom teaching, such as the *Mathematics Teacher* (volume 93, issue 8), *Educação e Matemática* [*Education and Mathematics*] (issue 27), and *For the Learning of Mathematics* (volume 11, issue 2). The ICMI (International Commission on Mathematics Instruction) study (Fauvel & van Maanen, 2000) also shows how determined the international community is about improving current classroom situations regarding the integration of history of mathematics into instruction.

Part II

How Can History of Mathematics Be Integrated into Classroom Teaching?

"There are as many different ways to integrate history of mathematics into classroom teaching as there are teachers" (Siu, 2000, p. 242). They depend on teachers' styles, beliefs, and preferences about historical topics. In particular, teachers' own personalities, enthusiasm, and willingness to leave their comfort and safety professional zones determine how appropriately they integrate history of mathematics into their classroom teaching, since such factors are actually critical to teachers' entire classroom practices (Bidwell, 1993; Furinghetti, 1997; Swafford, 1995). Some suggestions for historically inspired classroom teaching have been put forward. Based on her research, Furinghetti suggested that the integration of history of mathematics into the classroom may be aimed at developing mathematical knowledge or "at reflecting on mathematics" (p. 59). She also proposed a model for the general process of integrating history of mathematics into teaching which includes several stages, namely: (1) knowing the sources; (2) singling out topics suitable to the class; (3) analyzing the needs of the class; (4) planning the classroom activity, taking into account the availability of means, the aims, and the context of the activity; (5) accomplishing the project; and (6) evaluating the activity.

The last stage of Furinghetti's (1997) model deals with assessing the impact of an historical approach on mathematics education, which can only be done through qualitative analysis (Barbin, 2000), and it also refers to the assessment of student learning within such a learning environment. The variety of forms of classroom work that can be used as a means of integrating history of mathematics into mathematics instruction urges teachers to use alternative forms of assessment and, thus, to face their students' assessment differently. However, the history of mathematic *per se* should not be subject to examination (Fauvel, 1991), as an adequate integration of history of mathematics into classroom instruction asks this subject to be treated transversally, that is, throughout the entire curriculum and not as a curricular topic, isolated from all the others. This is exactly how the Portuguese reform documents and proposals for school mathematics have viewed the role of the history of mathematics into the various curricula (e.g., Abrantes et al., 1999; DES, 1997, 2002).

Implicit and explicit integration. The integration of history of mathematics into classroom teaching can be done either implicitly or explicitly. An implicit integration emphasizes "the redesigning, shortcutting, and signaling of … the path network that appeared historically and led to the modern form of" (Tzanakis & Arcavi, 2000, p. 210) mathematics. Therefore, the history of mathematics "is not an aim in itself, but a teaching itinerary is constructed … always keeping in mind the didactic aim" (Barbin & Menghini, 2000, p. 86). While in an implicit integration, the history of mathematics acts as a guide for designing instruction (van Ameron, 2001), in an explicit form, "the emphasis is on history" (Barbin & Menghini, p. 89). In fact, the overarching goal of an explicit integration of the history of mathematics into mathematics instruction is "to describe a historical period, to show the evolutions and the stages in the progress of mathematics" (Barbin & Menghini, p. 89). Barbin (2000) pointed out some dangers involved in explicitly integrating history of mathematics into classroom teaching. On one hand, students can develop "a false and truncated view of" (p. 65) the historical developments of mathematics if teachers' use of the history of mathematics is limited to providing students with piece-meal historical illustrations. On the other hand, there is the danger of running away from the goals of mathematics education towards an education in history of mathematics.

There have been some criticisms of the usual deductive ways in which textbooks present mathematical ideas (e.g., Avital, 1995; Davitt, 2000; Tzanakis & Thomaidis, 2000). Indeed, "a deductive (or even strictly axiomatic) organization of a mathematical discipline is given only after the discipline has reached maturity, so that it becomes necessary to give an *a posteriori* presentation of its logical structure and completeness" (Tzanakis & Thomaidis, p. 44). Moreover, such a deductive approach to mathematics may hinder students from realizing what really triggered its development. The history of mathematics can provide a means for intuitive approaches to teaching (e.g., Avital, 1995; Davitt, 2000; Tzanakis & Thomaidis, 2000). In fact, “similarly to what happened in history, instruction which moves from specific examples to theoretical generalizations will help the students make the leap to the higher level” (Avital, 1995 p. 6). However, teachers must also be aware that there may be some teaching approaches which are historically inspired, but which end up not helping students to develop their mathematical understanding, and even confusing them (Streefland, 1996).

Sources of historical material. Tzanakis and Arcavi (2000) distinguished three different types of sources of reference material. Primary or *original* sources concern excerpts from original documents, and *secondary* sources are comprised of history narratives, interpretations, reconstructions, etc. Finally, *didactical* sources refer to “the body of literature which is distilled from primary and secondary writings with the eye to an [instructional] approach … inspired by history” (p. 212). This is the type of source that is most lacking, and this is also the type of (ready-to-use) resources teachers most look for. Although studying an original source is very demanding and time-consuming (Jahnke, 2000), the reading of original work made by famous mathematicians constitutes “a simple and realistic way to introduce history of mathematics into teaching” (Barbin & Menghini, 2000, p. 88; Swetz, 1995b).

Original sources may be used either directly—texts are presented without any previous preparation—or indirectly—the sources are consulted following a motivating activity, such as solving a problem or giving a brief biographical account (Jahnke, 2000). In any case, teachers must carefully evaluate the interest and relevance of the chosen texts (Barbin, 1991). Many mathematics educators clearly support the use of original sources in the mathematics classroom for various reasons. Some argue that original sources help students see mathematics as a human activity, which improves their interest for learning (e.g., Barbin; Furinghetti, 1997), as such “sources can provide lively documental examples of genuine mathematical activity in the making” (Jahnke, p. 296). Others claim that, by working with original sources, students better understand and appreciate “the roles played by cultural and mathematical surroundings in the invention of new mathematics” (Laubenbacher & Pengelley, 1996, p. 257), as well as the processes of sophistication in concepts and techniques, and clarity and elegance in notation. Furthermore, students can compare the different approaches to the same concept and/or procedure made by different cultures and people in different time periods, and compare their own problem-solving strategies with those from the past, thus realizing “that there is no royal road to mathematical truth” (van Maanen, 1997, p. 46).

Secondary sources of historical material allow for the evaluation of old problem-

-solving techniques from the mathematical perspectives of today. For example, by looking at old books, students can analyze the mathematical content they address in comparison with what they find in their own current textbooks, besides appreciating the art in ancient printing and the simplicity and power of the current notation (e.g., Bruckheimer, Ofir, & Arcavi, 1995; van Maanen, 1997; Veloso, 1994). In addition, "old mathematical texts … provide insights into the culture and times within which they were written and give … hints as to the forces that shaped and controlled mathematical concerns" (Swetz, 2000a, p. 11). There are some risks in uncritically using secondary sources for historically-inspired mathematics instruction (e.g., Arcavi et al., 1982). For example, secondary sources of historical evidence frequently contradict each other (Gardiner, 1992; Katz, 1997; Rogers, 1991), and it is not difficult to find historical books that are indeed "historical fiction at its best" (Gardiner, 1992 p. 144), as some authors have no scruples in distorting historical facts.

Secondary sources of evidence easily lead to a number of historical myths and speculations that actually have little support, if any, in original sources (Gardiner, 1992; Heiede, 1996; Rogers, 1991). For instance, the idea that "Pythagoras went on bringing logic to mathematics" (Gardiner, 1992 p. 148) is one of the many priority claims that make no sense. Such risks can be substantially minimized by simultaneously using original and secondary sources of historical evidence, as original sources help the clarification and extension of what can be found in secondary sources, and they can "put in perspective some of the interpretations, value judgments, or even misrepresentations found in the literature" (Jahnke, 2000, p. 293). This is why didactical sources are important for a successful integration of history of mathematics into classroom teaching.

Many books and textbooks contain a number of false historical anecdotes (commonly believed as historical truths), and sometimes relevant historical information is ignored (e.g., Dennis, 2000; Heiede, 1996). For example, the association of Descartes with the typical curricular unit on graphing linear and quadratic equations, although common in textbooks, does not correspond to Descartes' contributions to the developments of mathematics. In fact, he "never used equations to plot points and create curves" (Dennis, 2000 p. 803). Rather than being associated with Descartes, the current approach to graphing linear and quadratic equations should be linked to Fermat, who proceeded from data or equations to plotting graphs as representations of those data or equations (Dennis).

Teachers, as well as curriculum developers, can find a wide variety of historical mathematical materials to start working with for classroom use (Furinghetti, 1997). Some of these materials are ready to use in the classroom (didactical sources, or what can be seen as classroom *resources*), but historical sources of evidence can also act as raw materials, to which teachers must always make the necessary adaptations to the needs, interests, and difficulties of their students (e.g., Estrada, 1993; Jahnke, 2000). It is this process of turning original sources (and secondary ones as well) into didactical (re)sources that is demanding and time consuming, and this is why teachers still claim they lack resources.

Some Ideas for Integrating History of Mathematics into Classroom Teaching

Teachers can integrate the history of mathematics into their classroom teaching at all grade levels, despite some disagreement about students' appropriate age levels and ability ranges (e.g., Fauvel, 1991; Gardner, 1991; Michalowicz, 2000). The vast majority of existing (ready-to-use) resources concern high school, secondary, and university education, and the few historical materials for elementary grades are costly and difficult to find. Some authors, though, argue that the history of mathematics itself is full of "elementary mathematics which can be used with surprisingly young children … [challenging the brighter students and simultaneously engaging] those with either mathematical or language problems" (Gardner, p. 19). There is a variety of forms for classroom work that can be used within a historical teaching approach, including biographies, birthday celebrations, visual displays (timelines, portraits, stamps, etc.), dramatizations, games, group projects and investigations (whose topics may vary from mathematical work to aspects of daily life such as pottery, mosaics, quilts, etc.), local historical explorations, historical homework problems, historical *problems of the week*, visits to museums, outdoor explorations of old instruments, videos, films, slides, internet surfing, and even entertainment shows (e.g., Barrow-Green, 1998; Carvalho et al., 1994; Fauvel, 1991; Hitchcock, 1992; Ponza, 1998; Swetz, 2000b; Tzanakis & Arcavi, 2000; Veloso, 1994). Teachers should use activities that call for cooperative work and student investigation, as these types of activities are appropriate for dealing with topics with which students are not familiar, and for stimulating mutual support and motivation (e.g., Fonseca, 1999; Johnson & Johnson, 1990). In particular, cooperative and investigative activities should allow students to do mathematics in ways that are different from those they are used to, and they should lead the students to realize the value of alternative solutions and the power of modern mathematics. In addition, teachers should encourage students to experience hands-on historical materials and the development of methods, tools, and mathematical ideas (e.g., Ofir, 1991; Reimer & Reimer, 1995), which stimulates their creativity and allows them "to experience an interchange between acceptance and rejection and the revision of preconceived ideas" (Thomaidis, 1991, p. 40).

The literature, particularly in teacher journals and magazines, provides teachers with many examples and suggestions of historically-inspired activities and investigations for classroom use. For example, we can find in Ernest (1998) several activities that deal with different mathematical areas such as fractions, Pascal's triangle, music, and magic squares. Wilson (2000) addresses four different areas of combinatorics, providing a brief historical account and a set of related classroom activities for each of those areas. Katz (1986) provides some examples of integrating the history of mathematics into the teaching of algorithms, logarithms, combinatorics, mathematical modeling, and trigonometry, and van Brummelen (1998) explains some activities about approximations of π and $\sin(1^\circ)$ that can be used on secondary mathematics classrooms. In the Portuguese journal Educação e Matemática, Relva (1993) suggested a classroom activity for first graders, based on local historical mathematical heritage, in which students study ancient number notations and computational techniques used in the Portuguese islands of Madeira mainly by fishermen and grocery sellers. Carvalho, Lobo, Milheiro, and Marques (1994) reported on a project involving over 20 teachers from different areas, about 250 students, and a few school staff members, who worked together towards the presentation of an entertainment show called *O bailado do* π [The ballet of π]. In addition to the suggestions from the APM journal and other accessible literature, Portuguese mathematics teachers can use the information and documents available through the GTHEM (APM, 2002) as well as the many sources they may have access to through the World Wide Web (e.g., Barrow-Green, 1998). The teachers' job, then, is to take these suggestions, and adapt and integrate them into their own classroom realities (e.g. Estrada, 1993). In the following lines, we will address in some depth a number of specific ways of integrating history of mathematics into classroom teaching.

Biographies and anecdotes. Using biographies of ancient and also contemporary mathematicians may be a good starting point for integrating history of mathematics into the mathematics classroom (Reimer & Reimer, 1995), as they bring "variety, color, and human interest" to the subject (Gardiner, 1992, p. 143). However, the use of such biographies should not be limited to small inclusions of historical curiosities, but should rather work as a springboard for mathematical investigations and explorations (Swetz, 1989). Investigating the lives and work of famous mathematicians enables students to see that the generation of mathematical ideas is a human and continuously evolving activity, and to have a fairly clear picture of the political, economical, cultural, and social contexts in which those mathematicians lived and those mathematical ideas were born and developed (e.g., Barbin, 1991; Estrada, 1993; Grugnetti, 1994; Shotsberger, 2000). The use of biographies may not be restricted to those of famous mathematicians. Indeed, there are many little known mathematicians whose work was also important in the development of mathematics (Pazwash & Mavrigian, 1986). The lives of other (famous) people may also lead to the integration of history of mathematics into teaching. For example, Perham & Perham (1995) give examples of "how mathematics can be used as an analytical tool" (p. 108) in non-usual areas. They use historical accounts about the life and voyages of the Portuguese navigator Fernão de Magalhães to make links to graph and game theories. However, the use of historical anecdotes can be prejudicial to the students if it is done in an uninformed and careless way (Gardiner; Rogers, 1991). Very often teachers who use historical anecdotes in their classrooms are led by their imaginations and run away from historical facts. Those same teachers tend to neither carefully choose the words they use nor double-check the information they provide their students (Gardiner). Furthermore, an over-reliance on anecdotal facts of the history of mathematics can deviate students' attention towards "interesting and curious references which are, in effect, not essential" (Grugnetti, 2000, p. 29), as they may distract students from the mathematical ideas they are expected to develop and better understand.

Historical problems. Problems and problem solving have been at the core of the historical development of mathematics (e.g., Barbin, 1996; Ernest, 1998; Swetz, 2000b), and they are central in current mathematics curricula and teaching and learning goals (e.g., Abrantes et al., 1999; DES, 1997, 2002; NCTM, 1994). Mathematics educators have looked for really good problems, "that is, problems whose solutions require application of certain mathematical concepts and techniques, whose contents demand a certain amount of interpretation, and whose presentation can capture and hold the interest of a student" (Swetz, 1986, p. 33). Such problems tend to generate other problems and to stimulate "mathematical explorations and classroom discussions" (Swetz, 1986, p. 33), whose success also depends on good accompanying questions (Thomaidis, 1991). Although today's recommendations for school mathematics ask for the solution of problems that are contemporarily relevant to the students, both teachers and curriculum developers can find a huge reservoir of good problems and pertinent questions in historical sources (e.g., Swetz, 1986, 2000b; Tzanakis & Thomaidis, 2000; Wilson & Chauvot, 2000).

Working on historical problems, or on historically-inspired problems, allows both teachers and students to understand the motivation behind the birth and development of many mathematical concepts and procedures. In fact, very often, what makes the beauty of a problem is not just its mathematical content but perhaps even more the reasons why the problem appeared (Tzanakis & Thomaidis, 2000; van Maanen, 1991). Students may engage more eagerly in working out centuries-

-old problems that triggered early mathematical developments (Swetz, 1989), than in doing the review exercises they usually find at the end of each chapter of their textbooks. Indeed, analyzing old ways of solving problems can lead both teachers and students to move forward from just doing mathematics to start thinking, and writing about it (e.g., Barry, 2000; Tzanakis & Arcavi, 2000; van Maanen, 1997).

Furthermore, historical problems from different time periods and cultural origins stimulate students to recognize and compare alternative problem-solving strategies (e.g., Grugnetti, 2000; Reimer & Reimer, 1995; Swetz, 1989), to establish mathematical connections (e.g., Grugnetti, 1994; Reimer & Reimer; Wilson & Chauvot, 2000), and to go further into their explorations and discussions (e.g., Fauvel, 1991; Wilson & Chauvot). In fact, both teachers and students are encouraged to develop more habits of research, including reading, looking for relevant information in libraries, and documenting findings. Truly historically-inspired activities should make students mathematical archeologists (e.g., Barry, 2000; Reimer & Reimer, 1995; Swetz, 1989, 2000b).

Dramatizations and ancient games. Dramatization is another form of integrating history of mathematics into classroom teaching. "Plays can be designed to re-experience the life of mathematicians in the past, as a way to appreciate the human side of mathematical activity" (Tzanakis & Arcavi, 2000, p. 229; see, for example, Ponza, 1998), or "to re-enact famous arguments in history, to let students revive … mathematical issues, as if they were their own" (Tzanakis & Arcavi, p. 230; see, for instance, Hitchcock, 1992, 1996). A dramatization, however, does not need to be a play (e.g., Horn et al., 2000; Shirley, 2000). A historical dramatization "can be as short as a guest appearance that lasts only a few minutes" (Shirley, p. 654), or it can be under the format of a dialogue or conversation. Dramatizations can be performed by the teacher, the students, others, or a combination of these. In any case, those involved are pushed towards doing some kind of research (Hitchcock, 1996; Shirley, 2000).

Playing ancient games stimulates the critical analysis of winning and losing strategies (e.g., Gorman, 1997). Typically, ancient games are simple to learn but "their challenging logic makes them almost addictive" (Gorman, p. 111).They can also be used for investigating the cultural background of the people who invented and played those games (Zaslavsky, 1991).

Local history and daily life. Often, sources of historical approaches to teaching can be found in aspects of daily life which people neither expect to reflect mathematics nor are aware of their mathematical characteristics (e.g., Wilson & Chauvot, 2000). Indeed, "an ethnographic approach to the history of mathematics aids a student's understanding and opens the door to an in-depth study of ethnomathematics"

(Marshall & Rich, 2000, p. 705). For example, through ethnographic historical investigations, teachers and students can find interesting relationships between the art of basketry and the golden ratio (Johnson, 1999), and between the marvelous tile decorations in Alhambra Palace (Spain) and many geometrical aspects (Brinkworth & Scott, 2000).

The history of mathematics includes the practical uses of the subject which "gain to be taken from the local history" (Veloso, 1994, p. 134; Fauvel, 1991). For example, several experiments have been carried out in Portugal addressing the mathematics underpinning the navigation techniques developed and used by Portuguese explorers during the 15th and 16th centuries. According to Veloso, astronomical navigation in the Atlantic Ocean began when Portuguese ships encountered strong headwinds close to the coast of Africa. The navigators' major problem was the lack of points of reference on land, since, in order to return to Portugal, they had to make long deviations from the coast to avoid the adverse winds. Studying how some navigation instruments (such as the quadrant and the astrolabe) worked provides students with opportunities for discovering and exploring the mathematics beneath the instruments' navigation rules so that they can better appreciate modern notation and mathematical language. Other activities involving those ancient navigation instruments to determine the width of a river or the height of a school building may also be of educational value in the classroom (e.g., Figueira, 2001).

Technology. The history of mathematics is also a source of technology-rich problems (e.g., Nagaoka & Isoda, 2000; Johnson, 1997). Indeed, the search for increasingly effective algorithms for finding the roots of equations, for instance, has also characterized the history of mathematics (e.g., Flusser, 1992). Currently, students have increasing access to computer technology, such as graphing calculators and computers, spreadsheets and dynamic geometry software. All these technological tools allow students to engage in technological explorations of historical mathematical topics, including, for example, Euler's and Newton's algorithms for solving equations (Flusser, 1992).

Conclusion

The list of different forms of classroom work addressed in this paper is not exhaustive, just like human imagination and creativity are unlimited. In our opinion, the many benefits of adequately integrating history of mathematics, in spite of a number of inherent difficulties, asks teachers to, at least, give it a try and engage seriously in pursuing such an instructional approach. We believe that, although the integration of history of mathematics into classroom instruction constitutes, with no doubt, a challenge to teachers' knowledge and feelings of security in the profession, ultimately, their sense of accomplishment and professional realization will also increase. Although changes will not follow immediately from integrating history of mathematics into classroom instruction, in the long-run, student motivation to learn and achievement will be improved (e.g., Fauvel, 1991; McLeod, 1992; Reimer & Reimer, 1995; Tzanakis & Arcavi, 2000). We agree with Fauvel (1991) when he urges teachers to understand "*how* the extra work which may be needed at first has a long-term pay-off in improving the attainment of" (p. 4) the goals of mathematics education.

Notes

[1] This paper was written under the scope of the coursework of the first author's Ph. D. program in Mathematics Education, at Illinois State University, Normal, IL, USA. Therefore, we had serious difficulties and limitations in terms of accessing documents pertaining to the Portuguese school realities and current reforms in mathematics education in general, and, in particular, pertaining to the history of mathematics and its educational use in Portugal. Nonetheless, we re-emphasize that this paper is not about the integration of history of mathematics into the Portuguese classrooms, but rather about the benefits for both teachers and students of such an adequate integration.

[2] All the listed publications are written in English. This fact does not seem to us to constitute an obstacle to the majority of Portuguese mathematics teachers as the texts are typically written in an easily comprehensible form, using clear language and simple writing styles.

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ABSTRACT. In this paper, we endeavor to sensitize mathematics teachers to the benefits, for them and their students, that result from an adequate integration of history of mathematics into classroom instruction. Acknowledging some objections and difficulties inherent to such an approach to curriculum, we also present a number of ways of integrating history of mathematics into classroom teaching that have been suggested in the literature. We provide several useful resources and references for helping and guiding classroom teachers in their attempts to integrate history of mathematics into their teaching.

Keywords: *History of mathematics, curriculum, classroom resources*

RESUMO. Neste artigo, pretendemos sensibilizar os professores de matemática para os benefícios, tanto para eles como para os seus alunos, que resultam de uma integração adequada da história da matemática no ensino da mesma. Cientes de algumas objecções e dificuldades inerentes a este processo de integração curricular, apresentamos algumas formas de integrar a história da matemática no ensino que têm sido sugeridas na literatura. Indicamos igualmente alguns recursos e referências que podem ajudar e guiar os professores a integrarem a história da matemática no ensino.

Palavras-chave: *História da matemática, Currículo, Recursos para a sala de aula*