

# **Approaching Early Algebra: Teachers' educational processes and classroom experiences**

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## **Structure of the paper**

This paper is devoted to the themes of the renewal of the teaching and learning of Algebra and it faces theoretical questions as well as aspects of the practice. We begin by giving a survey of the studies realized in the last 30 years, which have led to the birth of Early Algebra (§ 1). Then we present some theoretical constructs characterizing the theoretical frame of our experimental studies in Early Algebra (§ 2) and also concerning the teaching approach promoted in the classes (§ 3). We later sketch our methodology of work with teachers, aimed at supporting and monitoring the didactical processes enacted by them and, more in general, at refining the quality of their classroom actions (§ 4); finally we present an excerpt of a didactical process aimed at the construction of the sense of equations, paradigmatic as to the pupils' behaviours and productions and to the teacher's control and reflections (§ 5). We conclude with some considerations about the effectiveness of our methodology of work for the pupils' learning and for the teachers' professional development.

## **On Early Algebra**

The dawning of Early Algebra as a field of study, dates back to the second half of the Eighties, when some wide-scoped surveys on the difficulties of learning Algebra (see for instance Kieran, 1989) exposed the negative impact of traditional methods for the teaching of arithmetic, which were essentially based on aspects of calculus, paying little attention to the relational and structural aspects of arithmetic and the first studies about the possibilities to promote algebraic thinking in early grades appear (Davis, 1985).

In a critical review of Kieran's study published in Wagner e Kieran (1989), Both writes:

Students' difficulties in Algebra, it has generally been assumed, are largely difficulties in learning the syntax. Over the past decade, however, research evidence has been accumulating to indicate that many students have a very poor understanding of the relations and mathematical structures that are the basis of algebraic representation. This lack of understanding is not a new "algebraic" phenomenon: the research summarized by Kieran shows that the problem has its origin in arithmetic. Indeed, a major part of students' difficulties in Algebra stems precisely from their lack of understanding of arithmetical relations. The ability to work meaningfully in Algebra, and thereby handle the notational conventions with ease, requires that students first develop a semantic understanding of arithmetic. One task for research is to examine the whole question of students' recognition and use of structure and how this recognition may develop. A second task is to use this information to devise new learning activities and environments to assist students in this development. (p. 58)

Whilst in the same book, Herscovics highlights the fact that many cognitive obstacles are of a historical-epistemological or psychological nature. On the subject of the latter, he writes:

From the piagetian perspective, the acquisition of knowledge is a process involving a constant interaction between the learning subject and his or her environment. This process of equilibration involves not only assimilation — the integration of the things to be known into some existing cognitive structure— but also accomodation — changes in the learner's cognitive structure necessitated by the acquisition of new knowledge (p. 62). (...)

The obstacles associated with the learner's process of accomodation are pedagogically the most challenging. (...) What kind of pedagogical intervention can help the process along? (...) Problems need to be presented that can be understood by the learners but which cannot be solved within their existing knowledge (or at least not readily solved). Having created the need for change, the new material has to be organized into a constructivist teaching sequence, that is, a sequence starting from the learner's condition and expanding from it. (...) For any mathematical concept that is new to learners, the best we can do is create conditions likely to enable them to complete the difficult process of accomodation. There are, however, no guaranteed recipes. (p. 83)

Also to the same period belongs the debate on what may be, at curricular level, the ideal school age at which to introduce an initiation to Algebra. Authoritative scholars (e.g. Usiskin, 1987) maintain that this should happen in the eight grade, while others provide

documentation of curricular experiments conducted in their respective countries, with the introduction of Algebra in the sixth grade (Pegg & Redden, 1990).

Those years also saw the creation of interesting projects for the teaching of mathematics (in grades 6<sup>th</sup> to 10<sup>th</sup>), which promoted a constructive approach to Algebra from the very start, emphasizing the observation of relations and introducing the use of letters as a synthesis tool for analogous numerical relations or for the formal translation of verbal expressions which describe observed relations. Interpretative activities on formulae were also promoted, through comparisons between and reflections on different writings representing the same thing and similar writings having different meanings. There was an appreciation of methods based on reasoned attempts at the solution of equations and inequalities (see, for instance, Bell et al., 1985; Harper, 1987).

A significant milestone in this evolution process was the debate within the primary-school group of the working group on Algebra at ICME 7 (Quebec, 1992), co-ordinated by L. Linchevski (1995). In that context, it was emphasised that many of the difficulties pupils encounter in learning Algebra are caused by the mainly procedural teaching of arithmetic in primary school, which leads to an inevitable cognitive gap in the transition to Algebra, given the relational and structural aspects which are emphasised in Algebra and it was highlighted that

within primary-school arithmetic there is ample opportunity for the development of algebraic thought (p. 114)

and that

letters could be used within children arithmetic experience in order to facilitate their understanding of the meaning and significance of letters in later, formal Algebra (p. 114).

More particularly, with Sfard's theory of reification (1994) in the background, a new area in the teaching of arithmetic — called *Pre-algebra* — is conceived, with the aim of developing “pre-concepts” useful in Algebra, that is, complex arithmetical concepts of a structural type, representing the experience and conceptual basis into which to introduce more abstract and formal algebraic concepts, which can help overcome traditional “didactical cuts” (Fillooy 1990) or “cognitive gaps” (Herscovics & Linchevski, 1994).

In that context the group suggested a set of activities belonging to the pre-algebraic field, which included, among others:

- a) activities aimed at seeing the particular within the general and promoting generalization processes;
- b) the detection of analogies or differences within the structure of arithmetical expressions, through the analysis of the represented calculation processes and the highlighting of the role of brackets;
- c) the ingenious solution of equations, giving ample space to numerical substitution strategies and reasoned trials, aiming at the creation of appropriate cognitive schemes, through reflections on activated strategies;

- d) an introduction to the solution of verbal algebraic problems, through explorative procedures that bridge the gap between arithmetical and algebraic methods (for instance, those of “false position”).

In those years, several scholars (Arcavi, 1994; Arzarello et al., 1993, Gray & Tall, 1993; Filloy, 1990; Kaput, 1991; Lins, 1990) highlight the importance of students’ acquisition of what Arcavi names *symbol sense*<sup>1</sup>. They claim that students should mature their abilities, comprehension, and different ways of feeling through varied activities which lead them to acting with flexibility and instinctively in a given set of symbols, to moving through wider or different systems of symbols and to co-ordinating various interpretations of formulae in different solving worlds.

US starts proposing and debating the algebrization of the K-12 curriculum (Kaput, 1995, quoted by Carraher, 2001).

The second half of the Nineties saw a flourishing of studies on these aspects — both theoretical and experimental — and mainly targeted at pupils aged 11 to 13 (e.g. Anley 1999, Brito Lima & Da Rocha Falcao 1997, Da Rocha Falcão et al. 2000, Boulton Lewis et al. 1998, Charraher et al. 2000, Savadosky 1999). Some of the studies stand out for their theorising of models for a conceptual development in Algebra of a socio-constructive type, which highlights the influence of the classroom environment on teaching and promotes the use of physical means as tools of semeiotic mediation — all within the framework of an algebraic vision of Algebra as a language (e.g. Da Rocha Falcão, 1995; Meira, 1990, 1996; Radford & Grenier, 1996).

In an analytical study of students’ algebraic notations on the use of the scales as a mediation tool for making algebraic equivalences meaningful, Meira (1996) writes:

Using notations is obviously part of the very complex process of thinking algebraically (sometime an unnecessary part), but that indicates and supports the individual’ insertion in certain discursive practices that are critical for one’s participation and access to mathematics and, in particular, to Algebra. In this respect, using algebraic notations as part of a language connects the individual to “the spoken language of the mathematics classrooms” to “the use of particular words for mathematical ends”; to “the language of [mathematical] texts”; and also to “the language of written symbolic forms”. It is critical to note that this view does not limit Algebra to the use of its notational system, nor algebraic activity to the meaning intended by experts. The situation is similar to the young child that mumbles words in very simple sentences without completeness or syntactical correction, but plenty of meaning for the communication being attempted with an adult or a peer. (p. 378)

Later, Radford (2000), unifying within a wide theoretical framework historical-epistemological, psychological, semeiotic and didactic studies, goes as far as describing the learning of Algebra as

the appropriation of a new and specific mathematical way of acting and thinking which is dialectically interwoven with a novel use and production of signs whose meanings are acquired by the students as a result of their social immersion into mathematical activities (p. 241).

He sees the signs

as tools or prostheses of the mind to accomplish actions as required by the contextual activities in which the individuals engage (p. 241),

shifting the focus

from what signs represent to what they enable us to do. (p. 241)

Moreover, he underlines the key role of the teacher in establishing the mathematical practice in a social context.

The transposition of these conceptions from theory into practice — also motivated by the presence of Early Algebra in the school curricula of countries such as the UK (1991) and the USA (2000) — poses the problem of teachers' training and leads to the creation of specific innovation projects, as documented by the forum on Early Algebra within PME (2001) and more extensively by the Early Algebra section of the 12<sup>th</sup> ICMI Study, "The future of the teaching and learning of Algebra" (Chick et al., 2001).

To this framework belongs also our own *ArAl Project, Arithmetic pathways to favour pre-algebraic thinking* (cf. Malara & Navarra, 2001, 2003)<sup>2</sup>, a project born in 1998 on the basis of our previous studies at middle school level (Malara, 1994; Malara & Iadepola, 1999), and subsequently conceived for primary schools within the perspective of continuity between the two school levels.

## Our hypotheses and basic theoretical elements in the approach to Early Algebra

Algebra is usually introduced as a study of algebraic forms, privileging syntactic aspects, as if formal manipulation preceded meanings' understanding. As a consequence, algebraic language comes to lose some of its essential features: that of being a suitable language for describing reality, by coding knowledge or making hypotheses about phenomena; that of being a powerful reasoning and predicting instrument, that enables the individual to derive new pieces of knowledge about phenomena, by means of transformations allowed by arithmetic-algebraic formalism.

One first hypothesis of ours is that in the teaching of mathematics it is necessary to emphasize from the very beginning the representational aspects from which the mathematical discourse and mathematical knowledge develop. For clarifying our theoretical frame in Early Algebra, we shall introduce some basic constructs, which characterize our approach, precisely: 1. algebraic babbling; 2. process-vs-product and the question of rep-

resentation; 3. canonical and non canonical representations of natural numbers; 4. relational-vs-directional in the use of the equal sign; 5. Brioshi and the algebraic code.

### Algebraic babbling

Our fundamental hypothesis is that one cannot focus exclusively on *syntax* and leave *semantics* aside, but rather that there is a need to start from the latter, taking an approach to the teaching of algebraic language in analogy with the learning modalities of natural language. We make use of the *babbling* metaphor to explain this perspective.

In their early years children learn a language gradually, appropriating its terms in relation to the meanings they associate with them, and develop rules gradually, through imitation and adjustments, up to the school age, when they will learn to read and reflect on grammatical and syntactic aspects of language.

Our hypothesis is that the mental models that characterise algebraic thinking must be constructed since the early years of primary school, when children approach arithmetical thinking: this is the time to teach them to think arithmetic algebraically. In other words, algebraic thinking should be built up in children progressively, in a strict interrelation with arithmetic, starting from the meanings of the latter. Meanwhile, one should pursue the construction of an environment which can informally stimulate the autonomous processing of what we call *algebraic babbling*, i.e. the experimental and continuously re-defined mastering of a new language, in which the rules can gradually find their place within a teaching situation which is tolerant of initial, syntactically “shaky” moments. In this process, understanding the difference between the concepts of *representing* and *solving* represents a crucial point.

### Representing and solving: process and product

A very common pupils’ belief is that the solution to a verbal problem is essentially the statement of a result. This naturally implies that attention is focused on what produces that result: operations. Let us consider the following problem that poses a classical question: *There are 13 crows perched on a branch; other 9 crows arrive at the tree while 6 of the previous ones fly away. How many crows are now on the tree?*

Now let us to modify the question: *Represent in mathematical language the situation so that we can find the total number of crows.* Where is the difference between the two formulations?

In the first case, the focus is on the identification of the *product* (16), whereas the second concentrates on the identification of the process ( $13 + 9 - 6$ ), i.e. the representation of the relationships among the elements in play.

This difference is linked with one of the most important aspects of the *epistemological gap* between arithmetic and Algebra: whilst arithmetic requires an *immediate* search of a solution, on the contrary Algebra *postpones* the search of a solution and begins with a formal trans-positioning from the dominion of a natural language to a specific system of *representation*. If guided to overcome the worry of the result, each pupil reaches an upper level of thinking, substituting the calculations with the observation of him/her-

self reasoning. He/she passes to a meta-cognitive level, interpreting the *structure* of the problem.

### Canonical or non-canonical representation of a natural number

Among the infinite representations of a number, the *canonical* one is obviously the most popular. Thinking of a number means for anyone thinking of the *cardinality it represents*. But the canonical representation is *meaning-wise opaque*, as it says little about itself to the pupil. For instance: the writing '12' suggests a certain 'number of things', or at most the idea of 'evenness'. Other representations — always suiting pupils' age — may broaden the field of information about the number itself: ' $3 \times 4$ ' points out that it is a multiple of both 3 and 4; ' $2^2 \times 3$ ', that it is also a multiple of 2; ' $2 \times 2 \times 3$ ' leads to " $2 \times 6$ " and therefore to the multiple of 6;  $36/3$  or  $60/5$  that it is sub-multiple of other numbers and so forth.

We can say that each possible connotation of a number adds information to get to a deeper knowledge of it, as it happens with people: there are the first and family name, opaque if compared to other more expressive connotations of the subject, for instance with reference to other individuals he or she is linked to by social or family relationships (father of (...), teacher of (...), brother of somebody's husband). It is extremely important that pupils learn to see as appropriate the canonical representation of a number as well as any other arithmetical expression of which such number is the result (non-canonical representation of the number). In the case of twelve, appropriate and acceptable representations besides '12' are also ' $9 + 3$ ' or ' $2^2 \times 3$ '. This is done not only to favour acceptance and understanding of algebraic written expressions like ' $a + b$ ' or ' $x^2y$ ', but mainly to facilitate the identification of numerical relationships and their representation in general terms.

In relation to this, there is a delicate knot, e.g.: in  $15 \times (4 + 2) = 90$  a pupil, operating a reading left/right 'sees'  $15 \times (4 + 2)$  as an 'operation' and '90' as its 'result'. But he/she has to be educated to 'see' the sentence as an equality between two representations of the same number. The following paragraph is devoted to this aspect.

### The equal sign

In primary-school teaching of arithmetic, the equal sign essentially takes up the meaning of *directional operator*:  $4 + 6 = 10$  means to a pupil 'I add 4 and 6 and I find 10'. This is a dominant conception in the first seven or eight school years during which the equal sign is mainly characterised by a *space-time* connotation: it marks the steps of an operative simplification or reduction path (operations are carried out sequentially) which must be read from left to right up to its end (i.e. the reaching of the result). Later, when the pupil meets Algebra, the equal sign suddenly takes up a totally different, *relational* meaning. In a written expression like  $(a + 1)^2 = a^2 + 2a + 1$  it carries the idea of a symmetry between the expressions: it points to the fact that they represent the same number, whatever the value given to  $a$ , and the two expressions are said to be equal (in fact, they identify because they are equivalent with respect to the relation 'representing the same number, as

*a* varies'). Again, in the writing ' $8 + x = 2x - 5$ ' the equal sign points to the (*still unverified*) hypothesis about the equivalence of the two writings for some value of the variable  $x$ . Although nobody told him about this broader meaning, the student must now move into a completely different conceptual universe, where it is necessary to go *beyond* the familiar space-time connotation. But if the student thinks that 'the number after the equal sign is the *result*' he or she will be lost and will probably attach little meaning to a writing like ' $11 = n$ ', although he/she might be able to solve the linear equation leading to it.

These reasonings show an evident correlation to *linguistic* aspects such as the concepts of *interpretation*, *translation*, comparison of *paraphrases* and *conscious respect of rules*. In order to make pupils get aware of the role of such aspects, we introduced a fictional character called Brioshi.

### **Brioshi and the algebraic code**

Brioshi is a *virtual* Japanese pupil, aged variably depending on his interlocutors' age. He does not speak any other language except Japanese, but he knows how to use the mathematical language. Brioshi loves to find non-Japanese peers for an exchange of mathematical problems via e-mail. He was introduced in order to help pupils grasp the problem of the algebraic representation of relations or procedures expressed verbally and, above all, in order to convey the idea (difficult for pupils aged 8-14) that on using a language it is necessary to *respect its rules*, which is an even stronger need when the language is formalised, owing to the synthetic nature of the symbols used. Brioshi was introduced along with structured activities: an exchange of messages to be translated into mathematical language or natural language; where the 'expert' Japanese friend plays the role of controller of the translation. If Brioshi cannot understand the translation, this must be revised through collective discussion. Such 'role-play' works with all pupils, regardless of their age, and Brioshi's arbitral role, as a call to correctness and transparency, results very strong (for further details see Malara & Navarra, 2001).

So far we have reflected on mathematical and linguistic questions about Early Algebra, now we shall analyse the elements, which are at the basis of its methodological approach.

### **Relevant methodological aspects to approach Early Algebra in the classes**

The didactical situations we propose are born within stimulating teaching and learning environments, but they are not easily manageable by teachers. As a consequence, those who wish to undertake innovative educational practices need to deal with a set of relevant methodological and organisational aspects that actively support a *culture of change*. We shall now discuss some of these aspects in tune with the development of the class activities.



### The didactical contract

The didactical contract is a theoretical construct (Brousseau, 1988), which indicates the set of relationships, mainly implicit, that govern the pupils-teacher relationship when they face the development of knowledge concerning a particular mathematical content. These relationships make up a system of *obligations*, involving both the teacher and his/her pupils within the teaching and learning process, which should be fulfilled and for which each of them is responsible.

In the case of Early Algebra, with pupils aged 6-14, the contract concentrates on the construction of *mathematical conceptions* rather than of technical competencies. Pupils must be brought to the awareness of the essence of the contract: *they are protagonist in the collective construction of algebraic babbling*. This means they should be educated to gradually become sensitive towards complex forms of a new language, through a reflection on differences between and equivalences of meanings of mathematical written expressions, a gradual discovery of the use of letters instead of numbers, an understanding of the different meanings of the "equal sign", the infinite representations of a number, the meaningful identification of arithmetic properties and so forth. In this case, the didactical contract concerns the solution of algebraic problems and is characterised by the fundamental principle '*first represent and then solve*'. This seems to be a promising perspective when we need to face one of the most important key points of the conceptual field of Algebra: the transposition in terms of representations, from natural language (in which problems are formulated or described) to formal-algebraic language (in which the relationships they contain are translated).

### The interpretation of protocols

Protocols are written productions made by individuals or groups of students with reference to a task given by the teacher. In the case of activities aimed at the enactment of algebraic babbling, constructing competencies for the interpretation of protocols and a classification of translations made by pupils implies that the teacher has to face a variety of mathematical writings, often elaborated through a mixed and personal use of languages and symbols, linked to one another in more or less appropriate ways. Such a variety develops when the teacher stimulates, besides reflection, creativity.

When the pupils realise that they are producing mathematical thinking and contributing to a collective construction of knowledge and languages, they make a variety of mostly interesting and non-trivial proposals, which altogether represent a common legacy for the whole class.

The core of the activities, is the pupils' collective analysis and interpretation of the algebraic sentences they have produced. This interpretative work of their protocols is sharply intertwined with the practices of discussion in the classroom.

### Discussion on mathematical themes

By *mathematical discussion* we mean the net of interventions occurring in a class with reference to a certain situation on which pupils are requested by the teacher to express their

thinking and argue in relation to what other classmates expressed as well. Through this net of interventions, the situation is analysed and debated from different points of view, until shared solutions are obtained.

The enactment of a collective discussion on mathematical themes stresses on metacognitive and metalinguistic aspects: pupils are guided through a reflection on languages, knowledge and processes (like solving a problem, analysing a procedure), to relate to classmates' hypotheses and proposals, to compare and classify translations, evaluate their own beliefs, make motivated choices. In this context, the teacher should be aware of the risks and peculiarities of this teaching and learning mode.

The teacher plays a delicate role in orchestrating discussions. First, he/she must be clear aware of the constructive path along which pupils should be guided, and about the cognitive or psychological difficulties they might encounter. From a methodological point of view, he/she must try to harmonise the various voices in the class, inviting usually silent pupils to intervene, avoiding that leaders and their followers prevail and that rivalries between groups arise. Finally, he must help the class recognise what has been achieved as a result of a collective work involving everybody. He/she must learn to act as a participant-observer, that is to keep his/her own decisions under control during the discussion, trying to be neutral and proposing hypotheses, reasoning paths and deductions produced by either individuals or small groups. He/she must learn to predict pupils' reactions to the proposed situations and capture significant unpredicted interventions to open up new perspectives in the development of the ongoing construction.

This is a hard-to-achieve baggage of skills and a careful analysis of class processes is needed if a teacher wants to get to a productive management with pupils.

### **Our methodology of work in supporting the teachers' actions in the class and in promoting their professional development**

Our studies have always been realized in a strict co-operation with teachers and concern the design and experimentation of innovative didactical projects, in the frame of the Italian model of research for innovation (Arzarello & Bartolini Bussi, 1998; Malara, 2002). With time, our methodology of work has gradually become more and more refined. Though having its roots in the above-said model, it represents an important and complex evolution of the model itself. It fits in with the model of co-learning partnerships by Jaworski (2003), although it differs from it as to elements concerning the planning and realization of the teaching paths, the study of the students' learning, the relationship with the teachers and most of all the conceptions underlying the roles played by the partners<sup>3</sup>.

In tuning with the international trend (Sfard, 2005), in last few years our research has shifted towards the teachers, with the precise aim of finding out methodologies and tools that can promote their development about the mathematical/pedagogical competencies necessary to face a socio-constructive teaching.

Today, our studies are devoted to *the analysis of classroom processes* and have a double aim. On one hand, we want to offer to the involved teachers the possibility to have a more and more precise control on their own behavior and ways of communication and of observing the impact on classroom interactions of micro-variables linked to individual attitudes or to emotional-relational dynamics. On the other hand, we want to realize tools for the teachers education at large, to be used directly or in e-learning (about this last point see, for instance, Malara & Navarra, 2007).

These aims are pursued through the critical reflection on the transcripts of classroom processes, focusing on the interrelations between the knowledge built by students and the teacher's behavior in guiding students in their constructions. This process develops along four main phases: (1) teacher's autonomous reflection; (2) teacher and researcher's joint reflection; (3) teachers' common reflections; (4) teachers' reflection in interaction with the researchers.

(1). In the first phase, concerning the autonomous reflection on what happened in the classroom, teachers are asked to transcribe the recorded classroom discussions and to write explicit comments about the moments they consider problematic. This forces them to observe their action with detachment, in order to monitor the consequences of their ways to communicate with pupils, to ask questions, to give hints and to make decisions.

(2). In the second phase, after a careful reading of the teacher's transcripts, the researcher writes his/her line-by-line and general comments, then sends them by e-mail to the teacher. A joint analysis is done on a specific meeting between the teacher and the researcher. The researcher induces the teacher to make local reflections by asking him/her to explain the meaning/the reasons of some interventions, he/she indicates potential strategies for overcoming dead-ends and gives explanations about (sometime subtle) mathematical questions arisen. He/she also triggers global reflections on what has been done and objectifies significant steps in the development of the mathematical construction. This joint analysis provides an opportunity to make the teacher's habits, stereotypes, beliefs, misconceptions explicit and to disclose possible conceptualization gaps in his/her mathematics knowledge. This moment turns out to be of particular importance for the teacher's awareness of his/her way of being in class and for a first assessment of his/her decisions (didactical choices, interventions/silences, word turns to the pupils, reintroductions, timings, etc).

(3). The third phase, consisting of an exchange involving all the teachers who work on the same path in their classes, represents a moment of free sharing of the events, useful to express any possible fear or doubt, as well as to look for the roots of possible common questions. This phase also includes a cross reading of commented transcripts and other written reflections related to their classroom processes and an initial getting aware of the divergences of the individual action developments. This leads to further reflections on one's actions and to the formulation of some possible hypotheses and mainly to clarify aspects or questions to be discussed with the researchers. This phase is remarkable because it allows teachers to freely share beliefs and emotions and to feel protagonists of the process in which they are involved.

(4). In the fourth phase, the whole group reflection, a global revision of the teachers' transcripts is made and this turns out to be the climax of the whole experience. Sharing transcriptions and gathering the different classroom discussions arisen about the same problem situation, allows to spot out and objectify the reasons that have determined them. By comparing one's own developmental path with what colleagues did in the same steps of a teaching sequence, each teacher detects important distinctive elements and reflects on the effectiveness or limitations of his/her work (personal hasty and decisive interventions, little attention to listening, not understanding potentially fruitful interventions, scarce ability to orchestrate voices, difficulty in managing leaders or minimizing effects of tacit alliances, etc).

All this leads teachers to acquire deeper awareness of their way of being in the classroom, to better control their behaviour, to conceive new insights on their teaching and, possibly, to assume a new professionalism over a lengthy time.

### **An example of Early Algebra practice: A third-year class path on the ArAl Unit *From the Scales to the Equation***

Wishing to document the approach to Early Algebra according to our theoretical frame, we shall not focus here on aspects concerning our work with the teachers for their professional development, but we prefer to present an excerpt of a classroom process in a third grade class where it is possible observe the teacher' role in leading the pupils towards the construction of the equation and of its solution as representation of the solving process of a riddle. The focus shall be on the teacher's actions (didactical choices, her way to put herself to the pupils, reflections, emotions), and with the classroom culture (atmosphere, individual pupil contributions in the discussions, work in pairs or small groups). We shall also see, through the teacher's reflections, her own self-observation during the process, the impact of her emotions and convictions on her didactical choices, the establishment of new levels of awareness of the potential of discussion for the collective creation of meanings, and the importance of the meta-cognitive dimension either within the teaching-learning process or about the impact of the whole experience on the actors.

#### **The structure of the Unit**

The Unit, conceived for 10/11 year-old pupils, starts with the simulation of problematic situations on the scales, which are then solved by subtractions or splitting up of same quantities from both balance plates. By collectively reflecting on the actions taken to find a solution, students discover 'the principle of equilibrium' and the two principles of equivalence (see below). The problem then arises of how to represent the situations already examined. This phase involves the progressive simplification of the representation of the scales, slowly arriving at the equal sign and the choice of representation of unknown quantities, which leads to the 'discovery' of letters in mathematics and equations. Even the procedures for the solution of equations are progressively elaborated and refined through collective and individual activities, during which students elaborate

and compare various representations, refine their competence to translate sentences and, moreover, become accustomed to using letters as the unknown entity. A sequence of appropriately organized verbal problems of different levels of difficulty, leads students to investigating how to solve problems using Algebra.

### The class and the teacher

The class, made up of 22 pupils, is a middle-level one, including 5 high-level children, 5 at middle-to-low level, and 3 problematic cases.

The teacher tackles this experimentation with concerned uncertainty, since the Unit is graded for older pupils, and she is worried that division, being an operation which at a certain point gets in play, may not yet be sufficiently well grasped by the class. Inserted into a group with other colleagues engaged in the Scales Unit, attracted by her colleagues' achievements and *to avoid feeling isolated*, decides to tackle the Unit's initial part, also encouraged by the results she has already obtained in the class working on another ArAl Units.

### The teacher's adaptation of the Unit to the class

The teacher arranges the first part of the Unit to the class, planning all the activities foreseen in the *preliminary phase*, *action phase*, *representative phase*, *conclusive phase*, for a total of 13 hours. The activities of the different phases are separated by an average 5-day gap.

The preliminary phase is devoted to a discussion of equilibrium, aiming to assess the pupils' conceptions and check whether they already grasp the scales as a model of equilibrium. The aim of this phase is the objectivation of the general principle of the scales, namely that "the scales are in equilibrium if — and only if — the weights lying of their pans are equal".

The action phase concerns the collective exploration, with the virtual scales, of equilibrium situations (the so-called *Wizard's Riddles*), until a solution is found (4,5 hours). The aim of this phase is the objectivation as "theorems in action" (as intended by Vergnaud) of the two "principles of the scales": 1<sup>st</sup> Principle: "If the scales are in equilibrium, removing equal weights from their pans, the scales remain in equilibrium"; 2<sup>nd</sup> Principle: "With scales in equilibrium, if the weights on the pans are divided by the same number, the scales remain in equilibrium".

The representative phase is a complex one, and consists of specific activities geared towards the representation of: a) scales and equilibrium; b) unknown data; d) processes for the solution of the *Wizard's Riddles*.

The aims of the various stages of the activity are: a) the progressive simplification of representations of the scales' equilibrium, in search of a shared symbol to represent the equality of the weights lying on the pans; b) the introduction of symbols to indicate unknown quantities; c) the approach to equations, seen as an equality of different representations of same quantities and their solution through the codification of completed actions and of the principles that were implemented for their solution.

The conclusive phase is a checking phase of the pupils' learning and, more generally, a phase of wider reflection on what has been done. More specifically, it concerns the crea-

tion of riddles in pairs and their representation, on the model of the ones previously examined; also, the pupils answer a self-assessment questionnaire.

### Focus on a classroom scene

The scene in question refers to the representative phase of the path<sup>4</sup>. Of the various moments of this phase, though all of them very interesting, we wish to dwell on a significant episode concerning students' behaviours and teacher's meta-cognitive control of her own actions.

During the action phase in the collective discussion the children had tackled the solution of some riddles. We shall report about the first riddle: *We have on the scales: left-hand pan, a bag of flour and 50 g; right-hand pan, an 80 g weight. How can we find out how much a bag of flour weighs?*

During the representative phase, first the children had tackled collectively the representation of the equilibrium of the scales and had chosen, in a vote that followed the discussion, this symbol  $\frac{\quad}{\quad}$ , seen by the children as being the most representative of equilibrium. Then they had faced the representation of the riddle, with the instructions: *Try to tell what we did to solve the Wizard's Riddle with the language of mathematics, which uses numbers, signs, operation symbols, letters* and introduced, without any serious problems<sup>5</sup>, the letter to indicate the contents of a packet, even though with little awareness of its meaning (most children see it as an abbreviation of the name). Some of the representations produced by the children are reported below.

<p>Luca</p> $\frac{F + 50g}{F + 50 = 80}$ $\frac{F + \cancel{50} \quad   \quad \cancel{50} + 30}{F = 30g}$	<p>Laura</p> $\frac{F \cdot 50g = 80g}{F \cdot 50 = 80}$ $\frac{\cancel{F} \cdot \cancel{50} = \cancel{50} \cdot 30}{F \cdot 50 = 50 \cdot 30}$ $\begin{array}{c} \swarrow \quad \searrow \\ \quad 30 \\ = \end{array}$
<p>Giommi</p> $\frac{F \cdot 50g = 80g}{50g + F = 80g}$ $\downarrow$ $50 + (30g) = 80g$	<p>Greta B</p> $\frac{F \cdot 50g = 80g}{80 - 50 = 30}$

After observing this first activity, the teacher is *convinced that the children need help in structuring the representation*, and discusses again with them the representation of the 1st riddle, especially its central phase. This is the discussion:

Teacher: Children, the right passages have all arisen, now we want to put them into the right sequence. Many of you have written the right begin, the initial state of balance (she writes the representation on the blackboard)

$$\underline{F \quad 50g \quad = \quad 80g} \quad \textit{beginning}$$

Then you said that at the end the flour weighted 30 g. How can I write it with the language of mathematics? (*the final equality doesn't come from the pupils*)

Teacher (again): How did we write about salt in the previous riddle? What happens?

Giorgia: F is equal to 30g

Teacher: Good, let's record at the bottom:  $F = 30g$ ; the end. What about the middle part? Many have written:  $F + 50 = 80$ , but I remember taking 80 away and putting...

Luca: 50 and 30

Teacher: Right, so we have  $F + 50 = 50 + 30$ . At this point we need Marghe's sign, she had some crosses there

Giommi: Let's take 50 and 50 away

Teacher: How can we do it?

Lara: With a slash

Teacher: Ok, this is the first principle:  $F + \cancel{50} = \cancel{50} + 30$  that's why we need to see that we take away from both sides

Chiara B: Because even if you take away, the scales are in balance

This discussion is meaningful from the point of view of the teacher's approach. She acts as a model for the pupils showing them how they have to pose themselves in facing the task. In spite of the discussion's good results and the participation of the class, the teacher's reflection is very interesting:

I realise that, if I continue to conduct the representation collectively, I'll try with all means to lead the children to a writing that is "correct for me as a teacher", without taking into account their difficulties and logic and/or comprehension impediments, and without understanding who could actually reach a solution, though perhaps different from my own. I've decided that, from now on, I will help the children, but I will let operate individually, also to see their actual degree of involvement in the activity.

We regard this excerpt as very indicative of the calibre of the teacher, who not only manages to exert a constant control of her own behaviour in action, but also places herself on a level of “aims reflection”, examining her own actions in the context of the whole class process, and reflecting on their impact on the children’s behaviour and on the effects of their learning.

As examples of the productivity of this approach as to pupils’ interiorization of the experience, we show in the following table some riddles, with their corresponding representations, which the pupils “invented” for the Wizard.

Giommi & Davide	On the left-hand pan there are a packet of salt and bag of flour. On the right-hand pan there are a packet of salt and a 300 g piece. How much does the bag of flower weigh?	Fundamental Principle  1 <sup>st</sup> Principle	$\begin{array}{r} S, F = S \ 300g \\ \hline \cancel{S} + F = \cancel{S} + 300g \\ \\ F = 300g \end{array}$
Greta C, Laura & Chiara	On the left-hand pan there are a packet of little pasta stars and a 50 g weight. On the right-hand pan there is a 90g weight. How much does the packet of pasta stars weigh?	Fundamental Principle  1 <sup>st</sup> Principle	<p>Greta C</p> $\begin{array}{r} A \ 50g = 90g \\ \hline 90g - 50g = 40g \\ \quad \nearrow \\ \quad A \end{array}$ <p>Laura</p> $\begin{array}{r} A \ 50g = 90g \\ \hline A \ \cancel{50g} = \cancel{50g} \ 40g \\ A = 40g \end{array}$
Giulio & Renato <sup>6</sup>	On the right-hand pan there are 3 small packets of salt. On the left-hand pan there is a 600 g weight. How much is a small packet of salt worth?	Fundamental Principle  2 <sup>nd</sup> Principle	<p>Giulio</p> $\begin{array}{r} O.O.O. = 600 \\ \hline \cancel{200} + \cancel{200} + 200 = \\ = \cancel{200} + \cancel{200} + 200 \end{array}$ <p>Renato</p> $\begin{array}{r} 600 = SSS = 200 \ 200 \ 200 \\ \\ 200 + \cancel{200} + \cancel{200} = \\ = \cancel{200} + \cancel{200} + 200 \\ \\ 600 : 3 = 200 \end{array}$



As to pupils' learning, we can highlight their proper attitude towards the representation of the relationships between the involved quantities and their ability both to construct "naïve equations" by naming the unknown data and to represent the solving process. Of course, some questions on the syntactical transformations between additive and multiplicative representations (such as  $a + a = 2 \times a$ ) are still open and need to be refined. Moreover, pupils' attitude to make the equivalence principles explicit is particularly meaningful. However some pupils still have difficulties in representing and solving problems which require to refer to the second equivalence principle (see the footnote related to Giulio and Renato's protocol).

### The metacognitive dimension

*The pupils.* Here we present a self-assessment questionnaire handed out to the children at the end of this experiment in order to make them assess what they have done and learnt.

#### The Questionnaire:

WHAT TITLE WOULD YOU GIVE TO THIS ACTIVITY?

— DID YOU LIKE IT?

— WHICH PART DID YOU ENJOY MOST?

— WHAT WAS MOST DIFFICULT?

— WHAT DID LEARN? Here are some possible answers, but you may add others and choose the ones that are right for you:

— NOTHING

— TO COUNT

— TO REFLECT

— STRANGE THINGS, for example:

— TO TALK

— TO DISCUSS

— TO LISTEN TO OTHERS

— TO EXPRESS MY THOUGHTS

— TO WORK WITH NUMBERS AND LETTERS

— OTHERS...

As to the answers related to learning, many pupils declared that they learnt to reflect, to express their thought, to work with numbers and letters. Many others declared that they learnt to listen to others and to discuss. Only few pupils answered that they learnt to talk.

The answers given to the open question "I learnt strange things" concerned: thinking; expressing one's own thoughts with numbers; application and concentration; thinking with one's own head; deciding together with others; thinking collectively.

The teacher commented that these results seemed to her to be profitable and useful, both for the teaching of mathematics and for any other activity.

*The teacher.* We show here some excerpts of the teacher's final report on the impact of the experience on her and in the class. The teacher did a refined analysis, which highlights her awareness about the effects of the common work made according to our methodology on her own convictions and competencies. This shows us its efficacy for the teachers development.

The teacher distinguishes between the impact of the experience on the class and on herself. As to the class, she underlines different levels: the relational-social level, the disciplinary level and the level of the difficulties.

#### 1) Impact of the experience on the class

The experience, which was assessed by the children as decidedly positive, generated curiosity, interest, and, on the whole, a good degree of participation. The following could be some of the factors that had a positive influence: a) The diversity, compared with the daily routine, and the almost exclusively "oral" work, compared with the prevailing writing work, found in mathematics (so many columns operations!); b) The atmosphere of the work context, which was unique, involving, with all the children in a circle around a board, or at the blackboard, or at the scales: a bit of magic, where the Numbers Wizard launches challenges, which are accepted by the children. There are prizes, at first sweets, then a tag bearing the phrase "You're GREAT!", and, finally, the personal satisfaction derived from the solution of the problem. The activity produced an effect at several levels.

##### a) Relational-social level

The activity allowed all children to be involved, particularly those belonging to the middle and lower groups, who normally only tag along. There has been an activation of metacognitive and transversal capacities that are fundamental for communication and learning: attention, listening, reflection, expression in natural language of the children's own opinions, acceptance of criticism, collective decisions and even renunciation of personal positions.

##### b) Disciplinary level

The activity allowed the strengthening of some fundamental concepts of arithmetic: the number and its representation; operations linked to one another (simple expressions); equality; the letter as the "unknown entity" or as a "generic number". It has been possible to examine in more depth and strengthen the mathematical language, both natural and symbolic, in particular that of multiplication notation. I remember our discussion on the terms "double", "do twice", "multiply" (we saw the impact of the different meaning of the two terms in multiplication as seen as a repetitive addition and determined the importance of commutativity in putting them both on the same level).

c) Difficulties level

There emerged clear linguistic difficulties: for an eight-year-old child, it is still a great achievement to be able to correctly express thoughts, both as regards the semantics, given the lack of clear meanings (e.g. that of “division”), and the syntax, given the poor logical structuring of thought and phrase.

2) The impact of this experience on me as a teacher

The experimentation we carried out was extremely positive and strongly motivating from a professional viewpoint. It produced in me relapses and multiple effects. On the methodological level, the most important aspect concerned the strengthening of my capacity to: make decisions and micro decisions; be able to accept and tackle any deviation from a pre-ordained path; conduct a “real discussion” in the classroom, and analyse its components, observing the importance of my silent pauses, the relevance of the children’s contributions and silences; the preparatory and forecasting revision on the conversation; the conclusion shared by all, including the teacher (who would sometimes prefer different answers).

The recordings of the lessons made it possible for me to have a continuous feedback on my operating modalities, sometimes highlighting procedural mistakes, with missed or excessively prevaricating interventions, doubts on mathematical concepts, hesitations... .

On the theoretical-disciplinary level, the confrontation with teachers who are mathematically competent has allowed me to look in more depth at various arithmetical and algebraic themes, the latter being for me the more difficult ones.

### **Some closing considerations**

The complexity of our methodology entails an equal degree of complexity in the observation and analysis of the didactical phenomena taking place within it. Numerous are the variables to be taken into consideration, as also many are the viewpoints from which they can be examined in their reciprocal relationships: the pupils in the class dynamics, in their individual productions, in learning; the teachers in the classroom, with their colleagues, with their researcher contacts.

Here concentrated our attention on the teachers and their role in the development of a classroom culture in a pre-algebraic key. Through the study of one case, we have tried to show a cross-section of the intricate intertwining of operativity, study, reflection and comparison, which underlies our work with and for the teachers. It must be stressed that this is not an isolated case, but is actually typical of our operative standards (for an analysis of other cases, see Malara, 2003, 2005a,b; Malara et al., 2004; Malara & Navarra, 2005, 2007).

Of course, not all teachers display the same qualities as the one we observed here, of exemplary competence, her great attention to the involvement of the entire class, her awareness of what she is doing and her constant reflection on what is happening. In this regard, we should refer back to some of the considerations from our plenary at PME 27 (see Malara, 2003). Often, in the midst of live classroom action, teachers do not grasp pupil's reasoning, or fail to give it due recognition, thus allowing significant contributions to be dropped, or are conditioned by some pupils' invasiveness, or even unable to use appropriate silent pauses. All this shows us very clearly the importance of a fine teachers' education as regards reflection — both local, in relation to their action on the spot, and global, on the modalities and sense of their operating, so that they may rethink those occasions of missed interventions, of inappropriate decisions, and of how they might have been different and better.

An effective strategy for the growth of teachers in this respect appears to be the cross-comparison of parallel interventions. By comparing their own path with that of other colleagues along the same steps of an identical didactical sequence, teachers can reveal important elements of difference and reflect on the productivity or limitations of their work. This offers an opportunity to reflect on disclosed bad habits, on the underlying conceptions or emotions, leading them to the acquisition of a higher awareness of their way of being in the classroom, and to exert a greater degree of self-control of their attitudes and even to slowly modify them.

Globally speaking, after observation of the actors ensemble involved, we can confirm that in general:

As regards the teachers:

- An evolution of knowledge and a maturation of a new awareness, especially as concerns a critical re-reading of their basic knowledge, acquired almost exclusively by attendance at a (nearly always traditionally structured) higher education establishment or a non-mathematical university faculty;
- The refining of the sensibility in catching — at the very act of their arising — the potential of the pupils' contributions, their intuitions, and even the obstacles created by distorted or partial visions, very often extremely difficult to recognise, especially in pupils of this age.

As regards the pupils:

- The acquisition, through forms of algebraic babbling, of a *friendly* attitude towards the use of letters for codifying and generalizing observed facts;
- More generally, the achievement of a vision and appreciation of mathematics as a constructive discipline.

## Notes

1 Arcavi (1994) indicates the attitudes to stimulate in students to promote an appropriate vision of Algebra. Particular attitudes that he names include: the ability to know when to use symbols in the process of finding a solution to a problem and, conversely, when to abandon the use of symbols and to consider alternative (better) tools; the ability to see symbols as sense holders (in particular to regard equivalent symbolic expressions not as mere results, but as possible sources of new meanings); the ability to appreciate the elegance, the conciseness, the communicability and the power of symbols to represent and prove relationships.

2 The ArAl Project is led in collaboration with G. Navarra, a teacher-researcher who co-ordinates the organizational aspects of the Project and contributes to its scientific co-ordination.

3 A difference concerns the view of the teacher as researcher. Jaworski, like other scholars, for instance Breen (quoted by Peter-Koop, 2001), considers that a teacher cannot reach the quality of researcher. In Italy, instead, there is a long tradition of studies centered on the construction of this double-faceted figure, and today various teachers can be considered as full researchers both for the quality and autonomy of their research and for the international acknowledgement they have received.

4 A detailed analysis of the didactical process from which the excerpt is taken appears in Malara et al. (2004).

5 The children had already met letters, working on a previous ArAl Unit.

6 Observe, in Giulio and Renato's protocol, the difficulty which the pupils encounter in representing the solution process. There are several reasons for this, among them probably a lack of conceptualisation of division and of its associated operators, as well as the absence of adequate representation tools, but there is also the need to invent a means for overcoming, in the representation, the conflict between action (dynamic) and representation (static) and the objectivation — through appropriate signs — of the principles used.

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**Resumo.** Depois de um breve olhar sobre os principais passos que conduziram ao surgimento da *Early Algebra* como um campo específico de estudo, destacamos as vertentes principais do nosso trabalho e seu enquadramento teórico para promover o pensamento algébrico nos alunos e para orientar o professor na concepção e concretização de práticas favoráveis ao ensino da *Early Algebra*. Seguidamente, apresentamos um excerto de um processo didático, realizado numa turma do terceiro ano de escolaridade, que é significativo do ponto de vista das produções dos alunos e da cultura de sala de aula induzida pelo professor. Concluimos com algumas considerações sobre a eficácia da nossa metodologia de trabalho na aprendizagem dos alunos e no desenvolvimento profissional do professor.

*Palavras chave:* Tarefas que desenvolvem o pensamento algébrico; Pensamento algébrico; Generalização; *Early Algebra*; Desenvolvimento profissional.

**Abstract.** After a short survey of the main steps which lead to the birth of *Early Algebra* as a specific field of studies, we sketch the main features of our theoretical frame and our work to promote algebraic thinking in the students and to educate the teacher at putting into practice constructive paths of teaching of *Early Algebra*. We then present an excerpt of a didactical process, realized in a third-grade class, which is meaningful from the point of view of pupils' productions and of the classroom culture induced by the teacher. We conclude with some considerations about the effectiveness of our methodology of work for the pupils' learning and for the teachers' professional development.

*Key words:* Tasks that develop algebraic pensamento; Algebraic thinking; *Early Algebra*; Professional development.

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