

Developing algebraic reasoning: The role of sequenced tasks and teacher questions from the primary to the early secondary school levels

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Recent Algebra learning/teaching research has pointed to the importance of developing algebraic thinking for all students, from grades 2 to 12 (see Davis, 1995) — not just for secondary school students. The seeds for change in ideas about school Algebra were fuelled not only by the failure of students to develop either technical skills or conceptual meaning from purely skill-based approaches to Algebra instruction (e.g., Carry, Lewis, & Bernard, 1980; Küchemann, 1981) but also by socio-democratic principles applied to Algebra education (Chazan, 1996). Students had been found to experience great difficulty in making the transition from arithmetic to algebraic thinking — for instance, in moving from arithmetic operations and their results to working with variables and the generalized form of operations along with their transformations (see, e.g., Kieran, 1992, 2006, 2007). New ways of thinking about the content of secondary school Algebra, as well as the notion that introducing algebraic activity in primary school (Kaput, 1995), could help in overcoming some of the obstacles associated with secondary school Algebra, began to emerge. Algebra curricula were developed that included the concept of function as a central organizing theme (e.g., Heid, 1996), often supported by the use of multi-representation technology (Kieran & Yerushalmy, 2004) and by the motivating force of interesting problem situations (National Council of Teachers of Mathematics, 2000). However, the most far-reaching stimulus for change was perhaps the view that Algebra was not merely a set of procedures involving the letter-symbolic form, but also that it consisted of generalizing activity and provided a range of tools for representing the generality of mathematical relationships, patterns, and rules (e.g., Mason, 2005). Thus, Algebra came to be seen not merely as technique, but also as a way of thinking and reasoning about mathematical situations. Moreover, this latter perspective seemed amenable for inclusion at the primary school level, within a reconsidered approach to the teaching of arithmetic that drew on the inherently algebraic character of arithmetic.

This article begins first with a presentation of the various ways in which researchers describe algebraic reasoning in school mathematics, with particular focus on that of the primary school level. This leads into a discussion of the role of tasks and discussion questions in the development of students' algebraic reasoning. The rest of the article, which constitutes its main thrust, consists of examples drawn from the international research literature on algebraic reasoning that illustrate ways in which task and teacher questions

can be set up so as to encourage relational thinking, awareness of form, and generalizable approaches in students from the primary up through the early secondary levels. The task examples that are discussed all suggest the importance of one central feature: structured sequences of operations that draw students' attention to crucial aspects of form and its generalizability.

Conceptualizing algebraic reasoning

Carraher and Schliemann (2007), in their review of the newly-emerging research literature on the development of algebraic reasoning¹ at the primary school level, offer the following characterization of *algebraic reasoning*:

Algebraic reasoning refers to psychological processes involved in solving problems that mathematicians can easily express using algebraic notation. ... In the same sense that societies solved problems of Algebra before the existence of algebraic notation (Harper, 1987), students may be able to work with variables and the rules of arithmetic (i.e., the field axioms) before they have been taught Algebra. (p. 670)

Carraher and Schliemann's characterization thus stresses the implicit cognitive processes that may be at play among younger students when engaged in problem solving (such as, noticing structural relations and making generalizations) and suggests that some of these processes may involve variables and the rules of arithmetic.

Blanton and Kaput (2005) too emphasize generalizing processes in their characterization of algebraic reasoning: "We take algebraic reasoning to be a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (p. 413). They add, moreover, that algebraic reasoning can take various forms:

(a) the use of arithmetic as a domain for expressing and formalizing generalizations (generalized arithmetic); (b) generalized numerical patterns to describe functional relationships (functional thinking); (c) modeling as a domain for expressing and formalizing generalizations; and (d) generalizing about mathematical systems abstracted from computations and relations. (p. 413)

A still broader view of the nature of algebraic reasoning has been expressed by Lew (2004), a mathematics education researcher from Korea — a country where a formal approach to the teaching of Algebra with literal symbols begins in the seventh grade:

Algebra is a subject dealing with expressions with symbols and the extended numbers beyond the whole numbers in order to solve equations, to analyze functional relations, and to determine the structure of the rep-

representational system, which consists of expressions and relations. However, activities such as solving equations, analyzing functional relations and determining structure are not the purpose of Algebra, but tools for the modeling of real world phenomena and problem solving related to the various situations. Furthermore, Algebra is much more than a set of facts and techniques. It is *a way of thinking*. Success in Algebra depends on at least six kinds of mathematical thinking abilities as follows: generalization, abstraction, analytic thinking, dynamic thinking, modeling, and organization. (pp. 92–93)

In his remarks, Lew makes a distinction between Algebra as a set of facts and techniques and Algebra as a way of thinking. He also admits that the six ways of thinking that he describes are mathematical, that is, they go beyond the merely algebraic. In the Korean curriculum, the development of algebraic thinking at the elementary school level is based on the elaboration of activities related to these kinds of mathematical thinking.

These various views on that which constitutes algebraic reasoning at the primary school level are a reflection of the related views for the intermediate and secondary school levels. For example, Kieran (1996, 2004) categorized school Algebra according to the activities typically engaged in by students: generational activities, transformational activities, and global meta-level activities. While the *generational* activities involve the forming of the expressions and equations that are the objects of Algebra and the *transformational* activities typically deal with symbol manipulation procedures, the *global, meta-level* activities are somewhat special. These are the activities for which Algebra is used as a tool but which are not exclusive to Algebra. They include problem solving, modeling, noticing structure, studying change, generalizing, analyzing relationships, justifying, proving, and predicting — activities that could be engaged in without using any letter-symbolic Algebra at all. In fact, they suggest more general mathematical processes and activity. However, attempting to divorce these meta-level activities from Algebra removes any context or need that one might have for using Algebra. Indeed, the global meta-level activities are essential to the other activities of Algebra, in particular, to the meaning-building generational activities; otherwise all sense of purpose is lost.

The similarity between those activities that Kieran has named *global/meta-level* and Lew's characterization of the kinds of processes that form the core of algebraic thinking is quite striking. Even if the algebraic reasoning of older students presupposes symbolic objects and transformation techniques — objects and techniques that may be beyond the range of experience of younger students — the point here is that algebraic reasoning involves both general mathematical processes as well as ways of thinking about specific mathematical objects and operations that are quite distinct from, say, arithmetic thinking or geometric thinking or statistical thinking. Cuoco, Goldenberg, and Mark (1996) speak of an algebraic habit of mind that includes representing and extending; Mason (1996), generalizing and specializing; Driscoll (1999), algebraic modes of thought such as doing-undoing and abstracting from computation. Love (1986), who includes as well 'handling the as yet unknown,' emphasizes that, "becoming aware of these processes, and

in control of them, is what it means to think algebraically” (p. 49). But how does one go about helping students, especially younger ones, to develop such thinking? How does one, in the words of Mason (1996), “awaken this awareness”?

Promoting algebraic reasoning by appropriate tasks and classroom questions

Based on their research on the development of algebraic thinking in primary school, Carpenter, Franke, and Levi (2003) have argued that “*appropriately chosen tasks* can (a) provide a focus for students to articulate their ideas, (b) challenge students’ conceptions by providing different contexts in which they need to examine the positions they have staked out, and (c) provide a window on children’s thinking” (p. 14). However, they have also noted that, “although the selection of tasks can provide a context for engaging students in examining their conceptions of the meaning of [whatever object or transformation] (...), the *nature of the discussion of the mathematical ideas* is critical” (p. 18, emphasis added).

Most teachers rely, for their teaching of algebraic content, on the activities that they find in textbooks (e.g., Arbaugh & Brown, 2004; Kieran, 1992). However, the majority of textbooks offer little in the way of tasks for developing algebraic thinking, focusing as they do on manipulation. The more resourceful of Algebra teachers may turn to research journals or professional sources, but most do not have the time that is required to search out such resources to enhance the algebraic thinking of their students. Often, it is in the interstices of one textbook task and the next that teachers may attempt to knit together and offer to students some of the ideas related to algebraic thinking that have only been implicitly suggested by the textbook material.

Tasks and their relationship to learning have been a concern of mathematics educators for decades. Arbaugh and Brown (2004) remind us that Doyle (1983), for example, argued that “tasks form the basic treatment unit in classrooms” (p. 162) and that tasks “are defined by the answers students are required to produce and the routes that can be used to obtain these answers” (p. 161). Further, Arbaugh and Brown have described a set of criteria developed by the QUASAR Project (Stein, Smith, Henningsen, & Silver, 2000) for categorizing mathematical tasks on the basis of the types of thinking (i.e., levels of cognitive demand) that the task requires of students. Although the criteria do not relate specifically to tasks whose thrust is the development of algebraic thinking, they aim at supporting mathematics teachers in thinking about what constitutes worthwhile mathematical tasks. Among the cognitively higher-level tasks of the QUASAR categorization are those that “require students to explore and understand the nature of mathematical concepts, processes, or relationships” (Smith & Stein, 1998, p. 348). Smith (2004) adds that teachers and curriculum developers ought to be encouraged to use problems that go beyond practicing routine procedures, that is, “problems that help students to build mathematical connections and develop and apply mathematical concepts” (p. 96). The upcoming section of the article will attempt to unpack what it might mean

to “use problems that go beyond practicing routine procedures” in the context of algebraic thinking.

Hoyles (2001) has argued that, while much of the research that is presented in published articles does not say enough about the design of tasks, it is the teacher that shapes the learning that occurs not only through the organization of tasks and activities but also through the interactions that take place. Henningsen and Stein (1997) have similarly voiced the idea that classroom-based factors can shape students’ engagement with mathematical tasks in both positive and negative ways. Even tasks that have been set up to encourage high-level mathematical thinking and reasoning can be thwarted by the classroom culture. Most notably, “the extent to which a teacher is willing to let a student struggle with a difficult problem, the kinds of assistance that teachers typically provide students who are having difficulties, and the extent to which students are willing to persevere in their struggle to solve difficult problems” (Henningsen & Stein, 1997, p. 529) all tend to shape tasks and thereby influence the mathematical learning that occurs.

This article looks in particular at the kinds of questions that have been used in teacher- and/or researcher-generated tasks, and which have been geared specifically to fostering the development of algebraic reasoning, as well as at the discussion questions that teachers and/or researchers have posed to students as they grapple with new algebraic ideas. The three cases that follow below cover a range of ages and a variety of themes in algebraic thinking. In presenting these case examples, which are drawn from the research of three different researchers or research teams, I cite their work in greater detail than is normally done when referring to the research of others. I feel that this is necessary in order to capture the essence of their task sequences/questions and wish to express my appreciation of their tolerance in this regard.

Case 1: Thinking about equality in a relational way

The first case is drawn from the research of Carpenter, Franke, and Levi (2003) and deals with developing children’s conception of the equal sign and a relational view of number sentences. The thesis underlying the work of the Carpenter research team is that if students understand their arithmetic in such a way as to be able to explain and justify the properties they are using as they carry out calculations, they will have learned some of the critical foundations of Algebra. Tasks involving true/false and open number sentences (many drawn from the earlier work of Davis, 1964, in the Madison Project) were found, by the researchers, to be extremely effective, in that they:

- engaged students in discussions about the appropriate use of the equal sign;
- encouraged students to use relational thinking;
- fostered students’ reliance on fundamental mathematical properties when learning number facts, place value, and other basic arithmetic concepts; and
- helped students generate conjectures. (Carpenter et al., 2003, p. 134)

The example to be presented in this section made explicit use, according to Carpenter et al., of the following benchmarks. These benchmarks, which served as a guide to the construction of tasks, and their follow-up questions, related to the growth of children's conception of the equal sign:

1. Getting children to be specific about what they think the equal sign means represents a first step in changing their conceptions. In order for children to compare and contrast different conceptions, they need to be clear about what their conceptions are. This means getting beyond just comparing the different answers to a problem like $8 + 4 = \square + 5$.
2. The second benchmark is achieved when children first accept as true some number sentence that is not of the form $a + b = c$. It may be something like $8 = 5 + 3$, $8 = 8$. $3 + 5 = 8 + 0$, or $3 + 5 = 3 + 5$.
3. The third benchmark is achieved when children recognize that the equal sign represents a relation between two equal numbers. At this point they compare the two sides of the equal sign by carrying out the calculations on each side of the equal sign.
4. The fourth benchmark is achieved when children are able to compare the mathematical expressions without actually carrying out the calculations. (Carpenter et al., 2003, p. 19)

In this example (Carpenter et al., 2003, *Video Case 1.5*), a second-grade USA teacher was working with a group of 6 pupils (about 8 years of age). These pupils had all responded incorrectly earlier that week to the question as to the number that should go into the box of $8 + 4 = \square + 5$. Some had thought it should be 17; the others 12 — both answers indicating fundamental misconceptions of the meaning of the equal sign. The teacher then met with this group of students a couple of times a week over the next four weeks. For each task, she would ask pupils what they thought the answer should be and listened while they justified their own point of view or explained why they thought another person's answer was incorrect. For the first set of tasks, pupils were asked whether the following were True or False (note that each task was in a form that students would have seen before):

$$\begin{aligned} 3 + 5 &= 8 \\ 2 + 3 &= 7 \\ 58 + 123 &= 115 \\ 10 + 7 &= 17 \end{aligned}$$

The next set, which was designed to challenge them, contained the following tasks, which pupils were once again to decide whether they were *true* or *false*. Note that the first two below were non-standard equations for these pupils and were designed to get them to begin thinking about such uses of the equal sign. The third equation provided a basis for

beginning to talk about the fourth — an equation with two items on the right hand side, but one where the zero was intended to play as transition agent to other different equations with two elements on the right-hand side. The fifth equation was one, however, for which the pupils could not reach consensus.

$$\begin{aligned} 11 &= 3 + 8 \\ 8 &= 8 \\ 4 + 5 &= 9 \\ 4 + 5 &= 9 + 0 \\ 4 + 5 &= 9 + 1 \end{aligned}$$

A few days later, the following set was presented to the group. Note that the sequencing in these tasks was very important — from a standard equation, to one with 0 added to the right-hand element, to one where the order of these two right-hand elements was inverted, to one where it was not the total that was featured on the right side but rather the same decomposition as was featured on the left side. Clearly this sequence was intended to encourage awareness of both a “totaling” and a “comparison of elements on both sides” approach. Some pupils began to express as justification of the truth-value of the equations the fact that both sides had the same total or not.

$$\begin{aligned} 2 + 6 &= 8 \\ 2 + 6 &= 8 + 0 \\ 2 + 6 &= 0 + 8 \\ 2 + 6 &= 2 + 6 \\ 2 + 6 &= 6 + 2 \\ 2 + 6 &= 8 + 1 \end{aligned}$$

The next sequence of tasks included open-number sentences for which the pupils were to suggest which number would go in the box. Filling the box in the second equation below was quite straightforward: a 5 would create exactly the same equation as the previous one, which was true. In fact, the first equation also served as a basis for determining the truth-value of the 4th equation for one boy who stated that, if he took 1 from the first 4 and added it to the second 4, both sides would “look the same” (i.e., the $4 + 4$ being mentally manipulated to become $3 + 5$).

$$\begin{aligned} 3 + 5 &= 3 + 5 \\ 3 + 5 &= 3 + \square \\ 8 &= 8 \\ 3 + 5 &= 4 + 4 \end{aligned}$$

The last sequence included the examples seen below. In justifying their thinking, several pupils were now comparing both sides and doing mental movements of numbers, such as

those just described above, and thus suggesting an attainment of the 4th benchmark mentioned earlier. They had noticed new relations and adopted more powerful strategies. This thinking can of course be extended to dealing with equations such as $12 + 9 = 10 + 8 + c$ and even $345 + 576 = 342 + 574 + d$. Once students begin to realize that the equal sign signifies a relation between numbers, they have, according to Carpenter et al., the conceptual foundations for other aspects of algebraic thinking, such as properties, equivalence, and making sense of equation-solving transformations.

$$4 + 2 = 4 + 2$$

$$4 + 2 = \square + 3$$

$$4 + 2 = 3 + \square$$

$$5 + 3 = \square + 2$$

The crucial aspect of the tasks described in this case is their careful sequencing — a sequencing that was informed by the benchmarks listed above and that led students to question their conceptions of the use of the equal sign. Although I have described in detail the tasks presented by this particular segment of the Carpenter team's research project, space constraints do not permit an equally detailed accounting of the nature of the interventions and follow-up questions that were posed by the teacher in her interactions with this group of pupils. While the tasks themselves were extremely important to the growth of these pupils' understanding of equality and the use of the equal sign, so too was the way in which the teacher encouraged them to express their thinking and to try to justify it.

In summary, the algebraic thinking that was encouraged by the tasks described in the Carpenter et al. (2003) study consisted in thinking about arithmetic operations as relations between numbers rather than as computational problems. This in turn allowed students to treat the equal sign as other than a *do something* signal (Kieran, 1981), and to compare equalities with several terms on each side by decomposing or rearranging some of the number combinations, thereby demonstrating the numerical equilibrium of the equality. Students thus came to view number sentences in a manner that prefigured later work with algebraic equations.

Case 2: Quasi-variable thinking

The second case study on the development of algebraic thinking is drawn from the research of Fujii (2003) and his collaborator Stephens (2007). Fujii and Stephens have introduced young children to algebraic thinking through generalizable numerical expressions, using numbers as *quasi-variables* — for example, the -49 and the $+49$ in the number sentence $78 - 49 + 49 = 78$, which is true whatever number is taken away and then added back. Finding that “generalizable numerical expressions can assist children to identify and discuss algebraic generalizations long before they learn algebraic notation”

(p. I–59), Fujii (2003) claims that these expressions “allow teachers to build a bridge from existing arithmetic problems to opportunities for thinking algebraically without having to rely on prior knowledge of literal symbolic forms” (p. I–62). Fujii and Stephens (2001) have referred to this kind of algebraic thinking as *quasi-variable* thinking.

These two researchers have developed an elaborated task sequence based on the idea of quasi-variables, which they presented initially to pupils from grade 3 (8 and 9 years of age), and then extended to pupils in grades 5 to 8 (up to about 14 years of age), in Japan and Australia. The first part of the series of structured questions is rooted in the notion that, if one can convert a given subtraction into a one that involves subtracting by 10, the subtraction is made easier. Generalizing the process whereby this conversion is obtained is the theme of the task, which the researchers have called *Peter’s Method*.

In their research, the task began with the first set of questions below (Stephens, 2007).

Peter’s Method

Peter is subtracting 5 from some numbers. Peter says that these are quite easy to do. Do you agree?

$$37 - 5 = 32$$

$$59 - 5 = 54$$

$$86 - 5 = 81$$

But Peter says some others are not so easy, like:

$$32 - 5$$

$$53 - 5$$

$$84 - 5$$

Peter says, “I do these by first adding 5 and then subtracting 10, like $32 - 5 = 32 + 5 - 10$. Working it out this way is easier.”

Does Peter’s Method give the right answer?

Let’s look at the other two questions ($53 - 5$, and $84 - 5$). Can you use Peter’s Method on each of these?

Rewrite each question first using Peter’s Method, and then work out the answer.

Pupils who could use *Peter’s Method* to work out the last of the task examples were then asked to generate some examples of their own and to explain why the method works. They were subsequently requested to describe how they think Peter would have worked out the following: $73 - 6 = 73 + \square - 10$. If they could manage this, the task continued with the sequence displayed below.

*Peter says that his method also works for subtracting 7, and 8 and 9.
Can you show how Peter's Method works for these three questions:
Re-write each question first using Peter's Method, and then work out the answer.*

$$83 - 7,$$

$$123 - 8, \text{ and}$$

$$235 - 9.$$

Can you explain how this method always works?

Fujii and Stephens (Stephens, 2007; Fujii & Stephens, 2008) have pointed out that not all of the students who were interviewed in the grade 3 classes were able to apply Peter's Method; several needed to calculate each of the individual subtractions in order to arrive at the final difference and thus could not explain why *Peter's Method* always worked. However, Fujii and Stephens argue that tasks such as *Peter's Method*, just as was the case with the tasks used by Carpenter et al. (2003), offer an alternative for beginning to think about operations in a relational, rather than strictly computational, way.

In the second part of their study with students from the 5th to 8th grades, Fujii and Stephens first presented the *Peter's Method* task, and then followed it with extensions, which they called *Susan's Method*. *Susan's Method* involved larger numbers and some symbolic representations so as to encourage thinking about equivalent expressions and generalization of the patterns encountered in these expressions. The first extension of *Peter's Method* was as follows.

Susan said it this way:

"Instead of writing $32 - 5$, $32 - 6$, $32 - 7$, $32 - 8$ and so on, I decided to write the symbol ▼ to stand for the numbers 5, 6, 7, 8, and so on.

So, I wrote $32 - \text{▼}$ to represent all of these" (read as: "32 minus some number").

Susan then says:

"So instead of $32 - \text{▼}$ ("32 minus some number") Peter says $32 + \bullet - 10$ " (read as: "32 plus some other number minus 10").

Susan then says: "How does Peter find the value of the second number ●?"

What do these two numbers add up to?

What can you say about $\text{▼} + \bullet = ?$ "

"Could ▼ ("the first number") stand for a fraction like $7\frac{1}{2}$ or a decimal fraction like 5.2?"

Note that in the first extension of the task, the researchers used a symbolic representation for the various numbers being subtracted. In this regard, Fujii and Stephens insist on the importance of referring to the symbols ▼ and ● as "numbers" and *never* as "triangle" or "circle." In the second extension of the task, which follows, students were invited to extend the "subtraction of 10" method to the subtraction of 100, and beyond — but, first, without symbolic representations:

Can we look at how Peter's Method could be used for subtracting numbers like 95, 96, and 97?

Suppose Peter had $251 - 95$, what do you think he might do to make it easier?

What would he do if he had $251 - 96,5$?

What do you think he would do if he had $251 - 93\frac{1}{3}$?

A third extension of *Peter's Method* made use of symbols within a generalized relation involving the boundary condition of 100:

Remember what Susan did before. Now, instead of writing sentences like $251 - 95$, $251 - 96$, $251 - 97$, $251 - 98$, Susan again uses the symbol $\blacktriangledown$ to represent all of these.

What do you think she would write? (Pause for students to write $251 - \blacktriangledown$.)

Susan then re-writes this new sentence $251 - \blacktriangledown$ to show how Peter would subtract numbers like 95, 96, 97, 98 and so on.

She uses a second symbol $\bullet$ to write $251 + \bullet$

Can you complete this sentence?

How is the value of the second number $\bullet$ connected to the value of the first number $\blacktriangledown$?

What do these two numbers add up to?

What can you say about $\blacktriangledown + \bullet =$?

Could you use this reasoning to show how Peter would solve $251 - 83$?

In response to the last question of the 3rd extension, students remarked that the two numbers totaled 100, in other words, that $\blacktriangledown + \bullet = 100$. A few students even commented that *Peter's Method* could be applied to a wider range of subtractions that involved other than 10, 100, and even 1000. When asked how this might be expressed symbolically, one student from the 6th grade wrote " $\square - \bullet = \square + (\blacktriangle - \bullet) - \blacktriangle$ ", saying any subtraction could be converted into "an easier subtraction" by choosing a "cleaner" number $\blacktriangle$ (greater than the original number $\bullet$ being taken away) adding the difference $(\blacktriangle - \bullet)$ and then subtracting $\blacktriangle$ " (Stephens, 2007).

In designing this sequence of structured tasks, from the beginning of *Peter's Method* through its three extensions, Fujii and Stephens (see Stephens, 2007) applied the following considerations:

The first step in looking beyond particular number sentences to seeing generalizable patterns in these sentences is helping students to leave number sentences in uncalculated (unexecuted) form. (...)

A second step is to avoid premature generalization, (...) making sure that students work with exemplifications of the underlying general relationship. (...)

A *third* step in developing students' ideas of variable numbers is to acknowledge the importance to students of the boundary conditions implicit in *Peter's Method* and its three extensions [i.e., the 10 and the 100]. ...

A *fourth* step [in developing students' ideas of variable] is the use of representative symbolic 'terms' [such as ▼ and ●], as a way of summarizing multiple numerical expressions that students have already met.

In bringing to a close this second case related to tasks designed to encourage the development of algebraic thinking, I emphasize the concluding remarks made by Stephens in his 2007 paper: "The potentially algebraic nature of number sentences (...) can provide a strong bridge to the idea of variable; [but] it can also strengthen children's understanding of basic arithmetic." I might hasten to add, however, that this "strengthening of children's understanding of basic arithmetic" is not achieved by focusing exclusively on computation, but rather on students' observing patterns that involve the uncalculated arithmetic operations of special relationships and their becoming aware of the structure underlying these generalizable patterns or forms. While for the younger students of this study, the mathematical objects that were at the heart of their reasoning were numerical, the generalizing that they engaged in gave their reasoning an algebraic cast — even if they were not yet ready to symbolize the generalized objects as variables. In this sense, the mathematical process of generalization can be viewed as one of the bridges between the arithmetic and the algebraic worlds.

Case 3: Promoting algebraic reasoning by a focus on generalizable methods in the solving of word problems

The two previous cases have involved the use of sequenced, structured numerical tasks as vehicles for the development of algebraic thinking. The third case features word problems and the way in which a teacher's questions, if well conceived, can encourage students to make explicit their problem-solving approaches and to generalize them. This case is drawn from the research of Smith (2004).

Smith (2004) has argued that, despite the use of tasks designed to help students engage in classroom discussions that focus on making conjectures and reasoning mathematically, simply using such tasks will not spontaneously promote the desired discussions (see also Stein, Grover, & Henningsen, 1996). Teacher support is needed in order to help students engage in the reasoning that is intended by the tasks. In reflecting on the videos of the Eighth-Grade TIMSS Video Study (Stigler et al., 1999), Smith (2000) noted that, in contrast to the mathematical tasks presented by USA teachers where only 5 percent of tasks were implemented in a way that publicly discussed the mathematical relations inherent in them, 35 percent of the tasks deemed rich in mathematical relations were implemented by Japanese teachers in a way that promoted awareness of these relations.

The third case to be presented in this article focuses on the different ways in which two teachers, one from the USA and one from Japan, introduced and implemented similar problem situations. Smith (2004) consolidated her data for this research from the TIMSS Video findings (Stigler et al., 1999). Smith has described, as follows, the manner in which the USA teacher, Mrs. Jones, conceived of and presented the problem situation to her students. Mrs. Jones gave her students the *Prom Dress* task (see figure 1) because she thought it connected well with prior work done on writing equations and graphing lines and could be used to help students make a transition to systems of linear inequalities.

A few days ago, Veronica and Caroline were both asked to the prom. That night, they went out to shop for dresses. As they were flipping through the racks, they each found the perfect dress. Both dresses were priced at \$80. Neither of them had enough that night, but each went home and devised a savings plan to buy the dress. Veronica put \$20 aside that night and has been putting aside an additional \$5 a day since then. Caroline put aside \$8 the day after she saw the dress and has put in the same amount every day since. Today, their friend Heather asks each girl how much she has saved for the dress. She says, "Wow! Caroline has more money saved." How many days has it been since Veronica and Caroline began saving?

Figure 1 — The *Prom Dress* task (adapted from Smith, 2004)

When Mrs. Jones noticed that most of her students were having some difficulty in getting started with the problem, she tried to help them by drawing three labeled columns on the chalkboard. While students continued to work on the problem, Mrs. Jones encouraged those who had already finished to find another way to solve the problem. She later called upon various students to present their solutions. One student came forward and filled up the table of values on the chalkboard up to the seventh day, remarking that on the seventh day Caroline had more money saved than had Veronica. When Mrs. Jones asked if there were any other solutions, no one offered any and the “discussion” (and the lesson) ended. Mrs. Jones was disappointed that she had not been able to get her students to think of any other methods.

In contrast, Smith recounts how the Japanese teacher, Mrs. Hamada, was able to use a similar task (see figure 2) in a way that elicited the making of connections and the forging of more general strategies. Mrs. Hamada first put the chewing gum task on the board. A student read the problem aloud. Then Mrs. Hamada drew two rectangles on the board, one for Ken and the other for his younger brother. The students counted out 18 circles and displayed them on Ken’s rectangle to represent the 18 packages of gum, but counted them by tens to make clear the number of pieces of gum that Ken had started with. A similar process was followed for filling the brother’s rectangle and announcing the number of his pieces of gum. At that moment, Mrs. Hamada asked students to try to solve the problem.

Ken and his younger brother enjoy chewing gum. One day, the boys go to the candy store and buy several packages of gum. Ken bought 18 ten-piece packages of gum, and his brother bought 24 five-piece packages of gum. Every day, each boy finishes one whole package of gum. One day, they looked at how much gum each boy had left. Ken noticed that his brother had more pieces of gum than he had. How many days has it been since the boys bought the gum?

Figure 2 — The *Chewing Gum* task (adapted from Smith, 2004)

As Mrs. Hamada walked around the class to see which methods students were using, she encouraged them to explain their answers in a way that someone else might be able to understand what they had done. About halfway through the class period, she asked students to present their work to the class. The first group of students who came forward used a third rectangle and moved into it a ‘package of gum’ from Ken’s rectangle and a package from the younger brother’s rectangle, explaining that at the end of the first day, Ken had 170 pieces of gum and his brother had 115. They kept doing this until the younger brother had more pieces of gum left. Mrs. Hamada then summarized their approach: “You took one circle from each boy, counting down by tens for Ken and by fives for his brother until his brother had more gum. This is good, but it could take a long time when the numbers get bigger. Did anyone find an easier way than this?” (Smith, 2004, p. 101).

Another group came forward and drew a table of values on the board with three columns labeled: Day, Ken, and Brother. The values they entered into this table showed that on the 13th day, the younger brother had more gum. However, Mrs. Hamada did not stop there. She continued with the following:

Now I wonder if any of you thought of a way to show how many pieces of gum each boy had every day. Many of you may not have thought of this way we will do, but that is okay, we will try it anyway. I would like you to add some columns to Group 2’s table like this (headers: Day, Equation, Ken, Equation, Brother) and think of an equation Group 2 might have used to find out how many pieces of gum each boy had. What would Day 1 look like? (Smith, 2004, p. 102)

When one student from Group 2 responded that they took 10 away from 180 for Ken and 5 away from 120 for his younger brother, Mrs. Hamada filled this information in on the first line of the table ($180 - 10 = 170$; $120 - 5 = 115$) and asked the students to continue working on the task of completing the table. When some students appeared to be confused, she asked them to stop and look at the two numbers for a given day and to decide what computation they needed to do to get each number. If they found one way to do this, she asked them to think about whether there might be easier ways to do it. As the students continued working, Mrs. Hamada asked two students to put their work on the board (see figure 3).

Day	Student 1	Student 2	Ken	Student 1	Student 2	Brother
1	$180 - 10 = 170$	$180 - 10 = 170$	170	$120 - 5 = 115$	$120 - 5 = 115$	115
2	$170 - 10 = 160$	$180 - 20 = 160$	160	$115 - 5 = 110$	$120 - 10 = 110$	110
3	$160 - 10 = 150$	$180 - 30 = 150$	150	$110 - 5 = 105$	$120 - 15 = 105$	105
4	$150 - 10 = 140$	$180 - 40 = 140$	140	$105 - 5 = 100$	$120 - 20 = 100$	100
5	$140 - 10 = 130$	$180 - 50 = 130$	130	$100 - 5 = 95$	$120 - 25 = 95$	95
6	$130 - 10 = 120$	$180 - 60 = 120$	120	$95 - 5 = 90$	$120 - 30 = 90$	90
7	$120 - 10 = 110$	$180 - 70 = 110$	110	$90 - 5 = 85$	$120 - 35 = 85$	85
8	$110 - 10 = 100$	$180 - 80 = 100$	100	$85 - 5 = 80$	$120 - 40 = 80$	80
9	$100 - 10 = 90$	$180 - 90 = 90$	90	$80 - 5 = 75$	$120 - 45 = 75$	75
10	$90 - 10 = 80$	$180 - 100 = 80$	80	$75 - 5 = 70$	$120 - 50 = 70$	70
11	$80 - 10 = 70$	$180 - 110 = 70$	70	$70 - 5 = 65$	$120 - 55 = 65$	65
12	$70 - 10 = 60$	$180 - 120 = 60$	60	$65 - 5 = 60$	$120 - 60 = 60$	60
13	$60 - 10 = 50$	$180 - 130 = 50$	50	$60 - 5 = 55$	$120 - 65 = 55$	55

Figure 3 — The arithmetic equations produced by two students to yield the values in the Ken and Brother columns (drawn from Smith, 2004)

The students of the class were then asked which of the equation-types (that of Student 1 or that of Student 2) would be more helpful if the number of days got really large. They decided that Student 2's equation was the better of the two because it was more generalizable. They remarked: "All you need to know is how many days so you can multiply it by how many pieces of gum are in each package, ten or five" (Smith, 2004, p. 102). Because they then ran out of class time, the teacher concluded the lesson by asking the students to think about a more general way of writing the equation that would give them the number of pieces of gum each boy had on whatever day. One general formulation that they could possibly have generated is: (starting number of pieces) — (number of days) $\times$ (number of pieces chewed per day) = (number of remaining pieces).

Smith (2004) has pointed out that, while both teachers used similar problems to attempt to engage their students in the development of important mathematical ideas, there were significant differences in how they implemented the problem-solving situation and in how they orchestrated the conceptual extensions of student work. She has emphasized four of these differences.

- *The Japanese teacher modeled the problem situation, whereas the US teacher modeled a solution method.* The two teachers focused students' attention on different aspects of the problem. Mrs. Jones tried to help students get started by offering a possible solution method, (...) Mrs. Hamada helped students ground their understanding in the problem situation by helping them visualize the context and then asking them to find ways to resolve it.
- *The Japanese students gave more detail in their explanations. When this was not spontaneous, the Japanese teacher specifically probed students to give more detailed and connected explanations.* (...) Simply providing an answer was

not acceptable in Mrs. Hamada's class: If students did not make connections, she asked questions that linked the pieces together. (...)

- *The Japanese teacher helped students construct another solution method. ...* Mrs. Hamada was trying to get students to consider using a common variable as a first step to comparing the expressions. Similar to when she initially introduced the problem situation, Mrs. Hamada offered students a way to organize their information without directing students to use one particular method. (This could be seen when the two students constructed different ways to generate equations.) (...)

- *In the Japanese lesson, the solution methods presented were analyzed and compared. ...* Mrs. Jones's students presented only one solution method, allowing little room for developing mathematical connections across solution methods. Because Mrs. Hamada's students presented more than one solution method, she was able to have them highlight mathematical relationships. (Smith, 2004, pp. 104–105)

One of the features of Mrs. Hamada's implementation of the problem-solving task, which favored the evolution of students' algebraic thinking — the presentation of and comparison between two methods — has also been reported in other research on Algebra teaching. For example, in the book, *Connecting Mathematical Ideas* with its two accompanying CDs, by Boaler and Humphreys (2005), the ways in which Mrs. Humphreys' teaching encouraged the development of students' algebraic thinking has been captured in video-taped lessons and interviews of small groups of students from her classes. In one of these interviews, students speak about the importance, from their point of view, of having several students in the class present alternate solution methods:

Interviewer: Does the teacher mind when you have these different methods?

Student1: She loves it.

Student2: Because then we all understand it, instead of her telling us one — cuz not all of us are going to get it.

Student1: That way, if you don't understand one of them, there's a couple of others for you to.

Student2: Yeh, there's always, like three more, so you can get it. Like some people are visual, but then for other people that doesn't help them at all. (Boaler & Humphreys, 2005, Video-tape *Interview 1*)

These students stressed the value of seeing alternate methods as a means of *understanding* not only the problem situation itself, but also the connections between one approach for conceptualizing the problem and its solution and another. When problem situations are rich enough, there is an opportunity to include in classroom activity the discussion of

connections between representations, connections between solving approaches, and the encouragement of generalizable solving approaches. Nevertheless, such discussions do not automatically follow from the mere fact of using rich problem situations.

However, just as important as the pedagogical technique of comparing two methods — perhaps even more so with respect to developing algebraic reasoning — is the form used in representing the two methods in Mrs. Hamada's class: Sequences of operations that could lead to a generalization and the observation that one of the two sequences was more easily generalizable than the other. This was a key part of Mrs. Hamada's problem-solving lesson. Just as was the case with *Peter's Method* in the previous task sequence, neither the finding of a numerical solution to the problem being posed, nor the setting up of a typical algebraic equation to model the problem situation was the ultimate focus of the task activity. Rather the aim was to arrive at a set of operations that could suggest a generalizable method, which could be symbolized, but did not have to be — either with ▼ and ●, or with names of variables. Thus, the algebraic reasoning that was illustrated in these two problem-situation cases lay in thinking about and moving toward methods that could be generalized — In Case 2: generalizing the structure of the operations involving the boundary conditions and the quasi-variables; in Case 3: generalizing the relations among the changing values in the sequence of problem-solving operations and seeing the form that was implicitly suggested in that sequence.

Closing discussion

This article has presented three case studies of research designed to encourage the development of algebraic reasoning in students from the primary to the early secondary levels. Details of the sequences of task questions, as well as the follow-up questions posed by the teachers in group or classroom discussions, were the main focus of the case descriptions. What do these tasks tell us about the ways in which researchers and teachers seek to foster the growth of algebraic reasoning in students, especially those with little or no experience in symbolic Algebra?

The striking aspect of the tasks seen in all three of these cases was the importance given to fostering in students an awareness, or noticing, of a certain form in the given sequence of examples and of generalizing that form. As was pointed out by Stephens (2007), leaving the number sentences in an uncalculated format was crucial in the task and problem-solving sequences. While the mathematical process of generalization was common to all three cases, there were other processes at play that were not common. If we consider the mathematical content that was part of the thinking that students engaged in, some of the differences among the cases included modes of reasoning related to:

- Expanding the meaning of the equal sign to include equalities with operations on both sides — prefiguring the symmetric and transitive character of equality (Case 1: Carpenter et al., 2003);

- Developing a relational view of equality involving comparison of the two mathematical expressions without actually calculating — prefiguring the notion of a balanced algebraic equation and providing a meaningful basis for the later equation-solving transformation of performing the same operation on both sides of the equation (Case 1: Carpenter et al., 2003);
- Discerning special numerical relationships that were always true no matter what numbers, within certain boundary conditions, were used — treating numbers as quasi-variables that prefigured the use of algebraic variables (Case 2: Fujii & Stephens, 2008);
- Seeing sequences of solving operations in a general way — prefiguring the use of algebraic equations to model word-problem situations (Case 3: Smith, 2004).

Other examples of the kinds of algebraic reasoning that can be encouraged in students of various ages can be found in the research of, for instance, Blanton and Kaput (2005) with primary school students on the exploration of the properties of whole numbers, Radford (2006) with junior high school students on the generalization of figural patterns, and Kieran and Drijvers (2006) with secondary school students on some of the more theoretical and technical components of algebraic reasoning related to equivalence and factorization. As is suggested by the three references just cited, the nature of algebraic reasoning depends on the age and mathematical experience of the students, with the older students using letter-symbolic expressions/equations and algebraic transformations as the basic objects of their algebraic reasoning, rather than numbers and operations.

In closing, let me emphasize that for the three cases presented in this article, the specific instances of algebraic thinking that emerged in students over the course of the various research studies were all fostered by very-carefully developed task sequences and/or teacher/researcher questions. In that all these instances relied on generalization, it can without doubt be argued that generalizable sequences are a motor for, and an integral aspect of, the development of algebraic thinking, and that generalization has the power to raise students' thinking from particular numbers, particular operations, and particular problem-solving approaches to a higher level that prefigures variables, algebraic equations, and general solving methods. For the younger student not yet introduced to algebraic notation, these more general ways of thinking about numbers, operations, notations such as the equal sign, and problem-solving methods can indeed be considered algebraic. As Lew (2004) has remarked, "Algebra is much more than a set of facts and techniques; it is *a way of thinking*" (p. 93) — but algebraic thinking need not necessarily include "facts and techniques." The tasks presented in this article suggest that one of the principal ways in which such algebraic thinking can be fostered is by structured sequences of operations that draw students' attention to crucial aspects of form and its generalizability.

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Note

1 Note that the terms *algebraic reasoning* and *algebraic thinking* are used interchangeably throughout this article.

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Resumo. Este artigo começa com uma apresentação dos diversos modos usados pelos investigadores para descrever o raciocínio algébrico em Matemática, com especial ênfase no primeiro ciclo do ensino básico. A partir daí, discute-se o papel das tarefas e da discussão de questões colocadas pelo professor no desenvolvimento do raciocínio algébrico dos alunos. Segue-se a principal parte do artigo onde se apresentam exemplos recolhidos na investigação internacional sobre o raciocínio algébrico que ilustram de que forma as tarefas e as questões colocadas pelo professor podem ser apresentadas de modo a fomentar o pensamento relacional, a consciência da forma e abordagens generalizáveis nos alunos do primeiro ciclo ao início do secundário. Os exemplos de tarefas que são discutidos, na sua globalidade, sugerem a importância de um aspecto central: sequências estruturadas de operações que conduzem a atenção dos alunos para aspectos cruciais de forma e sua generalização.

Palavras chave: Tarefas que desenvolvem o raciocínio algébrico; Pensamento algébrico; Generalização; Pensamento relacional; Forma em *Early Algebra*; Métodos generalizáveis.

Abstract. This article begins first with a presentation of the various ways in which researchers describe algebraic reasoning in school mathematics, with particular focus on that of the primary school level. This leads into a discussion of the role of tasks and discussion questions in the development of students' algebraic reasoning. The rest of the article, which constitutes its main thrust, consists of examples drawn from the international research literature on algebraic reasoning that illustrate ways in which task and teacher questions can be set up so as to encourage relational thinking, awareness of form, and generalizable approaches in students from the primary up through the early secondary levels. The task examples that are discussed all suggest the importance of one central feature: structured sequences of operations that draw students' attention to crucial aspects of form and its generalizability.

Key words: Tasks that develop algebraic reasoning; Algebraic thinking; Generalization; Relational thinking; Form in early Algebra; Generalizable methods.

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