Reasoning reasonably in mathematics

John Mason
University of Oxford & Open University
Hélia Oliveira
Instituto de Educação da Universidade de Lisboa
Ana Maria Boavida
Escola Superior de Educação do Instituto Politécnico de Setúbal

Introduction

The word *rational* is often associated with reasoning and 'being reasonable'. Henri Poincaré (1956) expressed surprise that people find mathematics difficult to learn, because from his perspective it is entirely rational, and humans are rational beings. Jonathon Swift (1726) had however already challenged this notion, proposing that human beings are at best 'animals capable of reason'.

Teaching students to reason mathematically seems therefore to belong to a disputed domain between 'they do it already so it is only a matter of evoking and provoking it and drawing it to attention' and 'it requires specific and explicit teaching in order to come to students' minds when required'. Indeed many teachers would claim from their experience that it is difficult to persuade students to use various forms of reasoning such as reasoning by contradiction or by contrapositive. On the one hand very young children reason empirically (they pick up patterns in human and non-human behaviour and act as if they have internalised these); on the other hand they don't seem to know what to do when asked to justify something mathematically. The stance taken here is that there are delicate shifts of attention involved in reasoning 'reasonably'[1] in mathematics, and that teachers can be of considerable support if they are attuned to these different ways of attending.

In this paper we briefly describe two tasks that we conjecture can provoke young students to display mathematical reasoning, and report the responses to these tasks of some students in grades 4 and 7. We use the theoretical framework of structures of attention (Mason 2003) to highlight different aspects of student's reasoning.

Theoretical frameworks

We draw on the literature concerning the shifts in student thinking expected in mathematics in the primary years, a spectrum of development from explaining to yourself to justifying to others, and distinctions between ways of attending which are necessary in order to reason 'reasonably'.

Shifts in epistemology and justification

Hans Freudenthal (1991) observed that somewhere in primary school there is or can be an epistemological shift from the twin justifications of "it just is" and "a respected 'other' (adult) said so" (e.g. it is on the internet, or other media) to "it must be so because I can provide a reason". Such a shift is one of the concerns of this paper.

Of course providing reasons is fraught with difficulty. One difficulty is seeing the need for, and then providing the warrant for the 'reason' and how it pertains to the thing being explained. Another is chasing back the chain of reasons to some firm foundation. It is said that Diderot came across a copy of Euclid, and on opening it at some page became intrigued, finding himself following the chains of reasoning backwards, justifying the justifications, until he reached the axioms. School mathematics rarely establishes firm foundations, so there has to be some classroom norm or agreement about what is taken as 'intuitively clear' and what needs justifying. The tasks presented here are perhaps unusual in having clear 'foundations' as to what can be assumed and what not.

There is a time when many children become enamoured of asking 'why?' over and over. They soon discover that adults become agitated or exasperated. Children may perhaps be experiencing the mathematical-philosophical search for firm foundations which in mathematics connects with the Frege-Russell-Whitehead enterprise, and more generally with the social-science search for an objective method to provide certain knowledge. It may also be that children enjoy getting adults upset, and of course there may be a mixture of the two impulses. It is also possible that one of the ways in which children's reasoning powers are undermined is that adults give sequences of 'why?' short shrift, cutting off the 'game'.

A mathematical assertion of some generality may take one of three forms: 'something must be so' (with conditions specified), 'something cannot be so' (again with conditions specified), or 'it depends: it can be so but it can also not be so'. For example, consider a whole number n which is divisible by the number f: n may be the sum of f consecutive numbers (it depends); if f is odd, then n must be the sum of f consecutive numbers; if f is even then n cannot be the sum of f consecutive numbers.

Justifying and explaining

The essence of the epistemological shift from say-so to reasoning seems to lie in recognising that an assertion can have a different status in different contexts. It may be a conjecture, with weak or strongly suggestive evidence, or it may be justified. The justification may be adequate in one community but not in another. This is where the sequence of 'convince yourself, convince a friend, convince a sceptic' (Mason, Burton & Stacey 1982/2010) provides stepping stones from a 'sense of' why a conjecture is correct to a publicly acceptable justification.

In convincing yourself you look for explanations, for connections to what you already know, and perhaps to empirical evidence. In convincing a friend you try to bring to articulation the various connections and relationships that seem to have a bearing. In trying to convince a sceptic you attempt to meet all possible objections. Learning to be sceptical of other people's explanations is a valuable way to become better at convincing sceptics yourself.

Attention

The stance taken in this paper is that mathematical reasoning requires particular forms of attention. Not only must the justifier and the audience be attending to the same things but they must be attending in the same ways. Mason (2003, 2010) delineates three different macro structures and five different forms or microstructures of attention:

Macro Structure of Attention	*Micro Structure of Attention*
Focus: fuzzy or sharp	Holding wholes (gazing)
Scope: broad or narrow	Discerning details
Multiplicity: single or multiple	Recognising relationships in a particular situation
	Perceiving properties as being instantiated
	Reasoning on the basis of agreed properties

Holding wholes refers to the way in which the human perceptual system can 'take in' a scene as a 'whole', and this is often experienced in mathematics when gazing at a diagram without particularly distinguishing different component parts. *Discerning Details* refers to the act of discerning specifics, treating sub-wholes, which can be held or gazed at, and treated as entities. *Recognising relationships* refers to the search for relationships between discerned details as entities, which cannot be done unless those details have been distinguished. *Perceiving properties* involves a subtle but vital shift from dwelling in the particularities of relationships to perceiving these as instances of some more general relationship (property). *Reasoning on the basis of agreed properties* refers to formal reasoning using only agreed properties (axioms, assumptions). The claim is that experientially these five structures are very different states.

The three macro structures are important phenomenologically but difficult for an external observer to discern. Although the five forms of micro structure of attention were arrived at by an analytical route based on ancient psychology, there is a very close match between these forms of attending and the van Hiele levels (Burger & Shaunessy, 1986; Clements & Battista, 1991) which originally arose in the consideration of geometric reasoning (Dina Van Hiele – Geldof: see Fuys, Geddes & Tischler, 1985) and which were then applied more generally (van Hiele, 1986). The one major difference is that whereas van Hiele levels are usually seen as developmental states or phases which can be used to classify the thinking of students at different times, the five ways of attending can be experienced in quick succession and in no particular order.

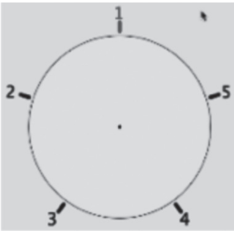
From the perspective of these five ways of attending, the core issue is whether students are encouraged and supported in shifting fluently between *recognising relationships* between discerned elements in a particular situation, and *perceiving properties* as being instantiated, because only then does it make sense to look for students to develop 'reasonable' mathematical reasoning based solely on 'agreed properties', that is, to results previously justified.

The tasks

The Secret Places task

The task "Where is the secret place?", originated by Tom O'Brien (2006), is to find a general strategy to locate the secret place in as few clicks as possible.

Usually, children quickly realise that it doesn't matter which number you click first, so it might as well always be place number 1. This is what mathematicians mean by the phrase 'without loss of generality', and is based on an underlying sense or awareness of symmetry.



Where is the secret place?

Five people are seated around a circular table. One of them is sitting in a secret place. What place is this? Try to find it by beginning to click on a number chosen by you. If it turns red, the secret place can be this place or it can be one of two adjacent places. Otherwise it will turn blue. The figure is an example. We clicked on place 1 and it turns red. So we know that the secret place is 1 or 2 or 5. Let's play!

As O'Brien demonstrates, when the task is presented in a manner suitable to the age group, even very young children naturally make use of case-by-case reasoning, and deductive chains of "if … then …", sometimes involving contradiction.

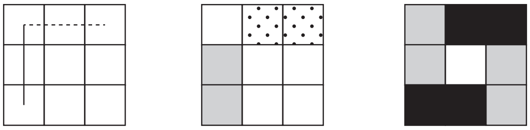
Although finding the minimum number of clicks needed for p places is on the edge of arithmetic, it only requires the same type of reasoning, so the task can be used with a wide range of ages. The applet has a variety of ways to extend the task, with more places at the table, more secret places, and extending the spread for turning red. There is even a two-dimensional version which affords access to the notion of nearness on different topological surfaces.

The Magic Square task

The aim of the task "Magic squares" is to engage students in thinking structurally (algebraically) without recourse to 'checking' with actual numbers.

> Magic squares
>
> A magic square is a square array of numbers with the property that the sum of the entries in each row, each column, and both main diagonals is the same. In figure 1 below the sum of the cells involving the solid line is

the same as the sum of the cells involving the dashed line, no matter what numbers appear in the 3 by 3 magic square. It follows then that in the figure 2 below the sum of the shaded cells equals the sum of the speckled cells.[2]



Figures 1, 2 and 3

> The task is to find other ways to shade and speckle cells so that the sum of the shaded cells equals the sum of the speckled cells and to justify your claim. As a challenge, use this experience to explain why, in the figure 3, the sum of the solids must be the same as the sum of the speckled cells.

As the aim of the task is to encourage students to think structurally, if they ask about how many actual magic squares there are, or how they can be constructed, then a separate task may be constructed and proposed to them. It is important to get students to reflect on the differences between constructing your own, constructing your own that you think will be hard to re-construct for others, and re-constructing one that you are given.

One possible extension for this task would be: "Find ways to shade and speckle cells in a 4 by 4 magic square so that the sum of the shaded squares equals the sum of the speckled cells. Construct one which is really hard to re-construct. Suggestion: start simply and build up a library of patterns".

Research question

Our research question was 'What sorts of reasoning would students make use of when invited to work on the Magic Square Reasoning task and the Secret Places task?'. Our analytic framework was the structures of attention, so we were looking for evidence of what different students attended to, and how. The conjecture to be investigated in this paper is whether from data collected about students work on tasks that call for justification, the notion of forms or structures of attention can shed light on what students are doing.

Secret places reasoning

Setting and method

This task was carried out during school time, but after the main classes were finished, in one room of the school equipped with a computer. The four students were organized in

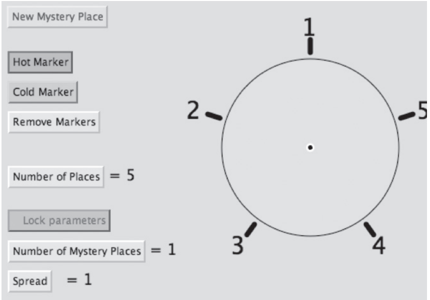


Figure 1

pairs: André and David; Isabel and Gabriela[3]. Each pair of students explored the task for about 30 minutes without the presence of the other peers. The teacher, Manuel and one researcher (one of the authors) were present to make notes about what was said and what else they noticed. The sessions were held on the same day and in sequence. The sessions with the two pairs of students were audio recorded and the records were transcribed. The examples of the students' thinking presented in this section were selected and translated from the above-mentioned transcription.

Students worked on the task in three phases: a period of familiarization, a period of work on the five place version (which is how the applet starts), and then a period of work on the six-place and seven-place versions, each with just one secret place to find.

First phase: Familiarization

The main goal of this phase was to help students to understand the background of the task and the functioning of the applet[4] (Mason, 2010a). The teacher began by showing each pair of students the computer screen as shown in figure 1[5] and explaining orally background information related to the task.

As he did so, he asked the students to click on numbers representing the people around the table in order to see if they understood what the appearance of the red and the blue colours meant. Then he suggested they make some trials to better understand how the applet works and how they could discover the secret place.

During this phase, students were not faced with any restriction about the number of clicks they could use. Furthermore, nothing was said that might decrease the level of demand of the task.

Second phase: To discover the secret place at a table with 5 people using as few clicks as possible

After being sure that students had no doubts about the meaning of the colours, either the teacher or the researcher told the students that they now faced the 'true challenge': "How to discover the secret place with the fewest possible number of clicks". This challenge was put orally, emphasizing the words "the fewest possible number clicks". The students had available a sheet of paper with the formulation of the task that they could consult if they so wished.

Immediately, students began to click on one of the numbers and to draw some conclusions about whether it was, or was not, the secret place. There were no difficulties concerning the functioning of the applet or the meaning of the colours. However, it was not always easy to get students to think before they acted in order to reduce the number of clicks. Furthermore, it was not always simple for students to express the reasoning that led them to make some decisions. Almost all the interventions made by the teacher or the researcher were intended exclusively to encourage reflection and explanation or justification of students' reasoning.

Third phase: To discover the secret place at a table with a number of places over 5 using as few clicks as possible

The teacher proposed to the students that they now consider six places at the table, and then later, seven places. In both cases, students were told that there was only one secret place. The students formulated conjectures more easily in the third phase than in the second phase and their reasoning spontaneously became more explicit.

Observations

First pair of students: André and David

André and David played eight games in total: two games in the first phase; three in the second phase; and three in the third phase (two with six places and one with seven places).

First phase: Familiarization

First game. They began by clicking on the number 5 and immediately after that they chose the number 1. Figure 2 shows the colours obtained. They then agreed that the secret place had to be 4. They justified this by saying that 5 is red and therefore the secret place could be 5, or 1 or 4. As 1 was blue, "this means that 1 cannot be". "It is 4 because 4 is the only one that remains adjacent to 5".

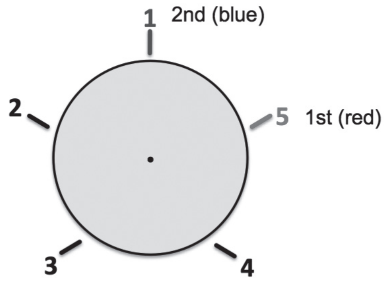


Figure 2

A very short time after, they started developing other hypotheses. However, they did not think, on their own, that the fact of a number being blue excludes the possibility of the two adjacent places be the secret place:

David: No, but if 4 is also red, [the secret place] can also be the 3 …

André: Or 5 …

David: Yeah, or 5.

Notice that their justification is partial: it is also necessary to mention that 5 cannot be the secret place because if it were, 1 would show red. Their attention seems to be focused on one place at a time, and on recognising the relationship between where the secret place is, and showing red when clicked. Perhaps this 'relationship' has not yet become a property to be instantiated several times, or perhaps their attention is single and narrowly focused.

The students started to reflect on the consequences of a number being blue only after being asked about which numbers would be red if 5 is the secret place (this question was correctly answered) and by the teacher emphasising that 1 is blue. From there, new conjectures immediately emerged about which number could be the secret place:

Teacher: See, 1 is blue ... (Figure 3)

David: So the secret place may be 3 or 4. 5 can no longer be.

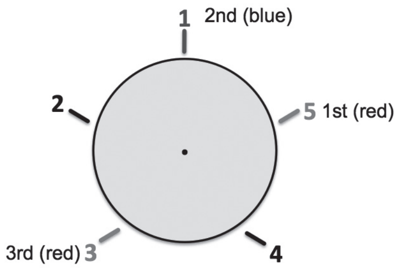


Figure 3

André: Yes, 5 can no longer be, because 1 is blue … (…) Or … 2 can also be [the secret place].

David: No, 2 cannot be … 1 is blue!

André: Oh… yes. It is true. But if this [points to 2] appears red, that [points to 3] can be [the secret place].

Teacher: Let us try …

David: If 2 appears red and 3 appears red, the secret place will be 3.
David clicks on the number 3 (Figure 3) and says: "As 5 and 3 turn red, it is 4". His colleague agrees.

The teacher and the researcher tried to expand students' thinking in order to understand whether they had 'forgotten' the idea that '3 can be the secret place' or whether they rejected this hypothesis intentionally due to 'good' reasons. We concluded that the rejection was not intentional:

David: It can still be 3 because we do not know the colour of the number 2.

Then David clicks on 2 and says: "It's blue! So it can only be 4". André agrees with his colleague's answer.

André's "But if …" again suggests a focus on one place at a time and then use of the property which specifies when a place shows red, as does David's "it can still be 3". It seems that attention is on the red-result, and then reasoning on the basis of knowing which places will turn red when clicked. They may not yet have appreciated the information provided by a blue result.

Second game. First, the students click on 3. They exclude this number as the secret place, "because it has turned blue". Next, they choose 1 and see that it turned red (Figure 4). At this point, and on their own initiative, they reasoned that there are certain numbers that cannot now be secret places, without needing to click on them, that is, they start to make inferences based on what they know about the rules of the game and the colours they get:

> DAVID: Can be 1 or 2 or 5 … No! 2 cannot be because of 3 that is blue.
>
> ANDRÉ: These two [numbers] can no longer be [points to 2 and to 3] (…) And 4 … 4 cannot be …
>
> DAVID: Yes, 4 cannot be. (…) So or it is 1 or 5. (…) Because of the 3 … is blue (…) For example, if this [points to 4] turns red, it cannot be [the secret place] because the 3 is blue.
>
> (…)
>
> ANDRÉ: And if 2 also come out red, it can no longer be [the secret place] because of this here [points to 3] that is not red.

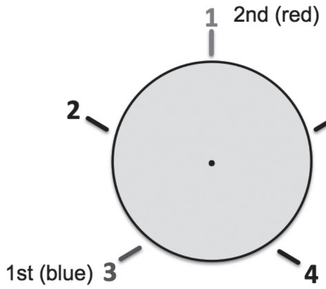


Figure 4

David begins to use multiple pieces of information, allowing his attention to be multiply focused, at least sequentially. What is not yet clear is whether, as is likely, their reasoning is based in the particularities of the relationships in this case, as distinct from being aware of the scope of generality of their reasoning. They seem immersed in the relationships in this situation, and are not yet perceiving these as instances of properties that hold more generally.

Having excluded, as secret places, the numbers 2, 3 and 4, one of them clicks on 2. He justifies his decision by saying: "It was to know whether it was red or blue … because if it was red, 1 was the only [secret place]". Since 2 is blue, the students quickly concluded that the secret place can only be 5. They don't pause to consider the consequences of it being red, so they are not yet perceiving the task as being about a strategy but rather locating the secret place in this instance.

Second phase: To discover the secret place with the fewest possible number of clicks

Third game. The exploration of each of the three games of this phase seems to have been greatly facilitated by the experience obtained earlier. For example, in the third game, the first of this phase (Figure 5), the students immediately excluded the numbers adjacent to a blue number as possible secret places, and they provided reasons for doing so.

This inference was maintained in all of the following games:

David: We're thinking … [the secret place], or it may be 2 …

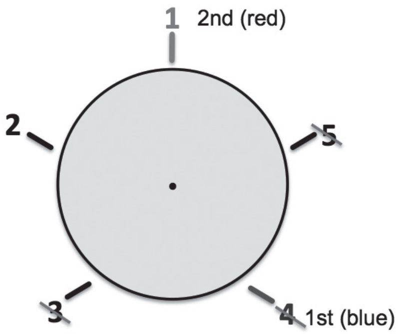


Figure 5

André: Or 1 …

David: Or 2, or 1 … yes, or 2 or 1. (…)

Teacher: Or 2 or 1. Why cannot 5 or 3 be the secret place?

David: Because 4 … is already blue. (…)

Teacher: How many numbers have you have eliminated as possible secret places?

André: Three. The 5 …

David: The 3 and the 4 … (…) Now we are sure that the secret place is 1 or is 2.

André: Well … (…) But if we clicked here [pointing to 3] and if it turns blue, we know that it is this [points to 1].

One of the students clicked on number 3, which turned blue. The immediate conclusion was that the secret place was at place 1. At this point they started to verbalize the following regularity: the circle must have three red numbers side-by-side. They stated that if 1 is the real secret place, as they think it is, 5 and 2 must turn red. They clicked on 2 and on 5 which both turned red.

There is clear evidence that they have internalised properties associated with blue and red places and are instantiating these in the particular configuration. It is interesting that they end up clicking all the places, in order to confirm their deductions. One reason that the applet never confirms a deduction is to try to encourage students to be satisfied by their reasoning without seeking external validation (from the machine or from an expert).

Fourth game. In this game the students used the regularity identified in the previous game to eliminate some numbers as secret places. Moreover, another strategy began to emerge for eliminating some numbers that could not be the secret place.

Figure 6 shows the first number selected and the corresponding colour. When they observed that 2 was red, David said at once that 4 cannot be the secret place:

Teacher - David, you said that 4 couldn't be the secret place. Why?

André: But it can be …

David: Becaus … the secret place can only be 1 or 3 …

André: Oh! Yes … because there are only three numbers and they must be side-by-side.

Teacher: But why did you say, so suddenly, that 4 cannot be?

David: Because … The 4 … If … If 4 turns red it is 3 …

Teacher: You are saying that if 4 turns red, 3 is the secret place? Is that it?

David: Yes. If 5 turns red, the place is 1. (…) And if 3 turns red …

André: The secret place is 2.

David: It is the 2 … This is the reason why 4 can no longer be [the secret place].

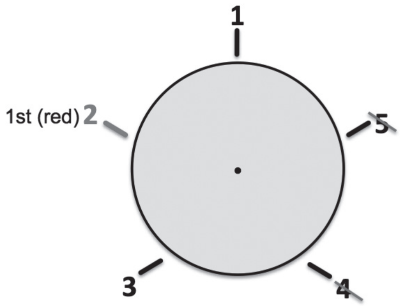


Figure 6

There are classic ambiguities in their speech which are truly proleptic, in the sense that everyone involved knows what is intended. For example in "If 4 turns red it is 3" the it referring to the secret place, and in "there are only three numbers" there is an unspoken taken as shared meaning that there are only three numbers which will show red when clicked.

It is possible that André, in his first quoted utterance, is thinking in general: in other situations it could indeed be 4, but then agrees that in this case it cannot be 4. Alternatively he hasn't yet applied his red-coloured reasoning when he makes his first utterance.

In this episode, David did not mention that 2 could also be the secret place, but he adds this information later. He seems to infer, more quickly than André, that 4 cannot be the secret place because, since number 2 is red, if 4 also turns red, the secret place must be 3 since the three red numbers must be side-by-side (be consecutive). Later, and for the same reasons, David excluded 5 as a possible secret place. His colleague does not immediately understand his exclusion:

David: And I also think that 5 cannot be the secret place.

André: No! I think 5 can be.

David: No … Because, look here [pointing to the computer screen] … No way!

André: Oh, you're right!

David: 5 cannot be. These two [points to 4 and 5] cannot be secret places.

In this game, students were very committed to using as few clicks as possible with the result that there was some unreflective thinking concerning the identification of the secret place. Sometimes, they had difficulties to express their reasoning clearly. For example, when students used the expression "this number can be", it was not always easy to understand whether they were referring to the colour red or to the possibility that the number could be the secret place:

André: Try the 1 here …

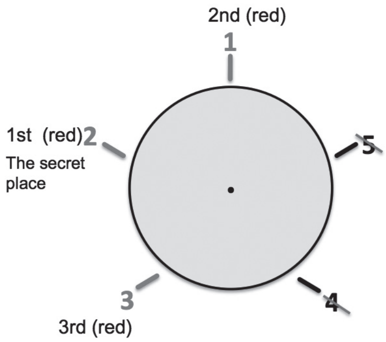


Figure 7

(David clicks on 1 that turns red)

David and André: It is the 2.

Teacher: Why is it the 2?

André: Because here … oh no! … it can also be the 5. (…)

Teacher: Why do you think that it can be 5, André?

André: I think it can be …

Teacher: Secret Place … we're talking about the secret place …

David: Yes, the secret place.

André: Ah, the secret place … I think that 5 cannot be.

David begins to focus on the minimum number of clicks needed to discover, with certainty, the location of the secret place. He begins by stating that the minimum is three, because there must be "three red numbers side-by-side". However, later he changes his opinion:

> David: There can also be only two [clicks]. Imagine that we choose 1 and 3 and the numbers turn red ... (...) The secret place can only be 2 because the three red numbers need to be side-by-side.

This is immediate evidence of reasoning by using a counter-example to disprove previous reasoning. He is also taking into account multiple pieces of information, and reasoning with them as agreed properties.

Fifth game. They began the third game by saying: "Let's try this". That is, they wanted to discover the secret place with just two clicks.

This suggests a conjecture that their desire had initiated some unreflective thinking concerning the identification of the secret place. We also noted this behaviour during the fourth game. Moreover, on some occasions, it seems that when André knew that a number could not be the secret place, he was sure that this number would show blue. That is, it seems that he consider that $A \Rightarrow B$ is equivalent to $\sim A \Rightarrow \sim B$.

Perhaps there is some slippage between showing red when clicked and being the secret place.

> Teacher: You clicked on two numbers, did you not? First you chose the number 3 which turned blue and after that you chose the number 5 which turned red (Figure 8).
>
> André: I know which is it [the secret place]. It is 1. (...) Because this [points to 4] can no longer be as 3 is blue and this [points to the 2] cannot also be because the 3 is blue. (...) The secret number must be between two red

numbers. (...) So 4 and 2 can no longer be because on this side is a blue number ... that's why it can only be the 1. (...) We have found it with only two clicks.

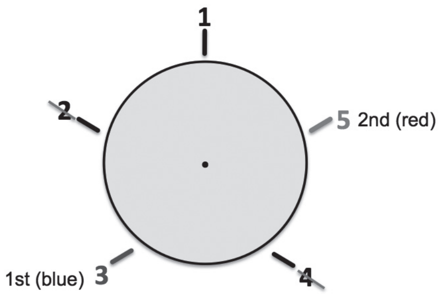


Figure 8

Again there may be some slippage between showing red and being the secret place. Alternatively, the excitement of the reasoning blocks out recognising other possibilities. In other words, attention might be focused on consequences but not on discerning the consequences on each place in turn.

In order for students to begin thinking that in this situation there could be other possibilities for the secret place (different from 1), it was felt necessary to intervene in order to focus their attention on the image on the computer screen, as well as on what they had said previously:

Teacher: So, you are saying that the secret place must be the number 1, because … What you were saying was that three red numbers must be side-by-side … Here is this blue number …

André: The red ones will be 5, 1 and 2

Teacher: The red numbers will be 5, 1 and 2 …

David: But, 5 may also be the secret number.

André: Oh, that is true … The number 5 can also be the secret one.

David: Imagine that the 2 turns blue … the 5 can also be because it is 1, 5, 4 …

Here David is imagining a possibility, transcending the need to click and test, and so opening the way to thinking more algorithmically, based on instantiating properties rather than recognising implications of relationships when clicked.

Third phase: A number of places greater than 5.

Sixth game. In this game, apparently, André, acts as if $A \Rightarrow B$ is equivalent to $\sim A \Rightarrow \sim B$. and quickly uses this claim to identify the secret place. His colleague's ideas led him to ponder his claim more deeply and to change his position:

Teacher: You clicked on 2 and it came out blue, did it not? [Figure 9]

David: 1 and 3 can no longer be. (…) Because 2 is blue …

André: It is the 5! [the secret place]

David: The 5, already?!!!

André: Because of ... look, 1 and 3 can no longer be … It's the same thing

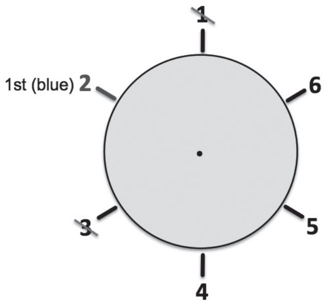


Figure 9

we have said before. In the centre is already a blue …

David: But, imagine … The secret place can also be the 4 … imagine the 6 is blue! These numbers side-by-side [points to 3, 4 and 5] can be red.

André: Oh! Yes … the 4 can be …

The need to consider each place in turn, and to apply the current information to it, is beginning to displace the desire to jump to immediate conclusions. Again it is associated with imagining and considering possibilities and how the properties provide information.

Seventh and eighth games. In the seventh game the idea begins to emerge that after we know where the place of one red number, a good strategy is to identify whether the remaining red numbers are on the right side or on the left side of the known number. This idea is used later on in the eighth game:

Example with six places (figure 10):

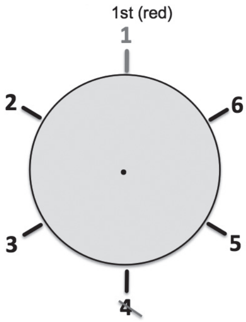


Figure 10

André: I think 4 cannot be. (…) Because the three red numbers … they must be side-by-side …

David: Yes, 4 cannot … it is like this or like that… [starting from 1, the student moved his finger to illustrate the clockwise hands' movement and the counter clockwise hands' movement]. We can never reach 4 (…) Because 4 is the number that is further away from 1 …

There is some as yet unspoken reasoning in order to justify the claim, but the notion of three red numbers side-by-side (consecutive) is beginning to emerge as a property to be applied rather than a relationship recognised in an earlier game.

Eighth game — seven places

David: 7 is red … so 3 and 4 can no longer be.

André: Yes.

Teacher: Why?

David: Because it is like this: three and three [starting from 7, he made the movements A and B, represented in figure 11]. Because 3 and 4 are the numbers that are further away from 7, and because the three red numbers must be side-by-side.

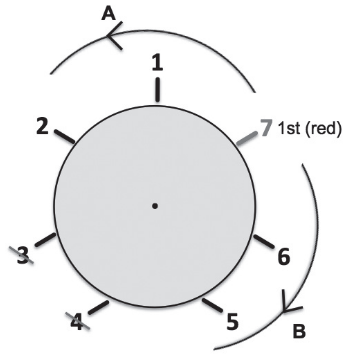


Figure 11

Note again the prolepsis associated with 'further': what seems to be intended is that they are 'further than 3 positions away from 7', as suggested by the movements of the fingers. But saying fully what you are seeing is not always easy to do. The property of the three reds being consecutive is now being manifested in gesture and in words.

The interactions of the pair of students give evidence of a gradual movement of an awareness (the consecutive property of the red places) from a relationship recognised in a particular situation to a property perceived as applying in many places. There is evidence of growing confidence, the use of gesture to instantiate a property and the beginnings of explicit articulation.

Second pair of students: Isabel and Gabriela

Isabel and Gabriela played six games in total, two in the first phase, one in the second and three in the third (one with six places and two with seven places). The activity of this pair was generally similar to the activity of the first group so it will not be described with so much detail. Our focus now is on the main differences observed between the two groups.

One of these differences is that Isabel and Gabriela arrived at some conclusions quickly and with more autonomy than their colleagues. For example, in the first game they quickly conjectured that if a number turns blue, the adjacent numbers cannot be secret places. After some trials, they justified this conjecture. The justification process was similar to the one used by David and André:

> Gabriela: The 2 cannot be because of 3.
>
> Teacher: 2 cannot be. What is it about 3?
>
> Gabriela: It's blue.
>
> Teacher: And then?
>
> Gabriela: If 2 is the secret place, 3 must be red, but 3 is blue … So 2 cannot be (…) And 5 cannot also be because the next is 4 and 4 is blue.

They seem to have turned the relationships associated with a blue-place into a property that can be applied in various situations. This makes it possible for them to use this information in their reasoning. Their utterances display a degree of prolepsis similar to that of the first pair of students.

The property 'the three red numbers must be side-by-side' and its use emerged early in the third game (second phase). The decision to move to the third phase was made by the students after playing only one game during the second phase. Under their own initiative it was also decided to move to a table with seven places after playing just one game with six places. This suggests a sense of confidence, of having a good grasp of how to find the secret place, even though they had not formulated an algorithm or tried to minimise the number of clicks necessary.

Figure 12 shows the three first choices made by the students in the fourth game (the first game of the third phase) and the order of these choices.

The interchange presented below reveals that one of the students (Gabriela) conjectured that the three blue numbers must also be side-by-side. Her colleague did not seem very convinced and tried to discover the secret place using only on the property concerning the red numbers:

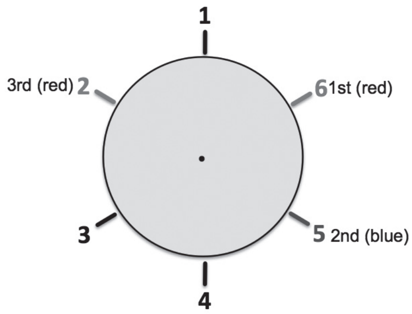


Figure 12

Gabriela: First, I clicked on 6 and it turned red and then on 5 and it turned blue. I think the red numbers must be side-by-side and the blue numbers must also be side-by-side. If 6 is red and 5 is blue, 2 and 1 can be the secret place. 4 and 5 can no longer be because 5 is blue. They will be blue.

Teacher: Isabel, I would like to hear you...

Isabel: 5 cannot be the secret place because it is blue. So, to have the three numbers side-by-side, as we saw in the other game, the possibilities [to the secret places] that I still see is 2 or 1.

Teacher: 2 or 1 … We don't know yet …

(…)

Isabel: Now I chose 2 because then we can see if the red numbers would look like this [she moves her hand in the counter clockwise way, starting from 6] or on this side [she moves her hand in the clockwise way, starting from 4].

Teacher: So, what number do you think is the secret place?

Gabriela: It's 1.

Teacher: What do you think, Isabel?

(…)

Isabel: Gabriela said she thought the blues had to be side-by-side and that the 4 had to be blue. But there is also the possibility of having the three red numbers here [pointing to the numbers 6, 1 and 2] or here [pointing to the numbers 3, 2 and 1].

Gabriela: But if so, then there are two secret places! ...

Teacher: There is only one secret place.

In her last utterance Gabriela displays deductive reasoning followed by contradiction in order to show that Isabel's conjecture does not hold. There is a lovely ambiguity that often arises, as people confuse the colour displayed when clicking, and being the secret place. This is evident when Gabriela says that since 6 is red and 5 is blue, that 2 can be the secret place. Place 2 is too far from 6 to make 6 blue. Indeed, by the 'three reds side-by-side' reasoning 1 must be the secret place.

Later, Gabriela displays a mix of deductive and contradiction reasoning in order to justify why place 1 is the secret place and why the blue numbers must also be in a row:

Gabriela: I think it cannot be as Isabel said, because there are six places and half of six is three. So if the numbers are red and blue, three are red and three are blue. Because the secret place is red and it has a red number on each side. The reds must be side-by-side. I think the secret place is 1 because 6, 1 and 2 are together. 3, 4 and 5 are also together but 5 is blue and 3 and 4 will turn blue … 2 and 6 are red and 1 will also be red.

Here the proleptic "the reds must be side-by-side" refers to the places that will display red when clicked.

This reasoning is different from any other reasoning made by the students of the other groups. Isabel seems to be convinced by the argument of her colleague. Indeed, during the next game (the fifth) Isabel tried to discover the secret place by using the property that 'the three red numbers must be side-by-side and the three blue numbers must also be side-by-side'. However, during this game the students struggle with a question: Can the number of red numbers be greater than 3? Eventually they exclude this possibility based on the fact that there is only one secret place:

Isabel: Now there are seven places. As seven is an odd number, it cannot be half … It must be four side-by-side or three. Red or blue …

Teacher: You are considering the number of red numbers and blue numbers? How many are red?

Gabriela: Yes. Three or four.

Isabel: Three? … Because … to be side-by-side …

(…)

Gabriela: I think that it can only be three red numbers because there is only one secret place. If they were three blue numbers the other four will be red. There are seven places. If there were four red numbers, we must have two secret places.

Teacher: Isabel, I do not know what do you think about this …

Isabel: I agree with Gabriela.

Here there is another clear example of reasoning by contradiction, a form of reasoning often thought to be too difficult even for senior high-school students.

Next, the students began to try to identify a strategy that allows them to figure out quickly where the three red numbers and the four blue numbers might be in the circle. Figure 14 shows the first four choices made by the students during the fifth game. The following interchange indicates why they choose the numbers and why they have followed the order shown:

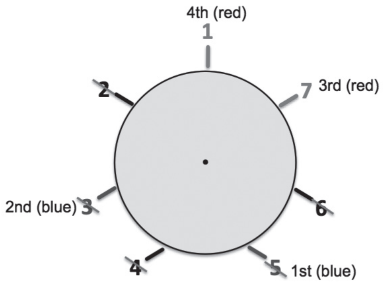


Figure 13

Teacher: Isabel, why at this point (3rd click) did you choose 7?

Isabel: I have always been jumping a number to see if they turned blue or red … I was thinking that when I found a red one, I would click on the two numbers that are on each side of the red number. But now ... I do not know … 6 can no longer be the secret place because the number that is next to it is 5 … it is blue.

Teacher: So where do you think the secret place may be? There is only one secret place …

Gabriela: I think the secret place is 1.

Teacher: Why?

Isabel: Because 6 cannot be because it is next to 5 which is blue; 4 cannot be because it is next to two blue numbers; 2 cannot be because it is next to a blue …

Gabriela: But it [the secret place] could be 7 …

Isabel: And also 1. 1 is adjacent to 2 and we have not clicked yet on 2 …

Teacher: What possibilities do we have for the secret place?

Isabel and Gabriela: Either 1 or 7.

The interaction reveals that Isabel may have developed a strategy of clicking on a number that was next but one to a place which already had a blue colour, but next to a number which had a red colour. Their attention seemed to be on finding a red place. Using this strategy, the students were able to eliminate some numbers. However, they realised that the clicks up to that point were not sufficient to identify the secret place. They hadn't fully internalised the information yielded by a click in order to discover that clicking on 6 or 2 would give them the information they wanted. There is a residual desire to click on where you think the secret place might be, despite the fact that this is often less informative than clicking elsewhere.

They then decided to make a fourth click. However, they noted that it was still not possible to find out where was the secret place:

> ISABEL: We have to click on another number … On 1. It turns red …
>
> (…)
>
> GABRIELA: 1 and 7 may be also … [the secret place]
>
> ISABEL: I think we now have to click on 2 or on 6 to see which number can be released (…) I think we still have to spend one more click.

A click on 2 or 6 would have enabled them to deduce the location of the secret place. Clicking on 1 contradicts Isabel's apparent current strategy and is less informative than clicking on 2 or 6.

At the end of the fifth game, the students were unhappy because they felt they had used too many clicks to discover the secret place. This may be why in the last game they changed their strategy. The interaction presented below illustrates how they conducted this game, but since they got a red immediately, a different part of their strategy came into play:

> GABRIELA: 3 may be [the secret place] because it is red (Figure 15).
>
> TEACHER: Now what is the next number you will choose?
>
> ISABEL: 2. If 2 turns blue, 3 cannot be [the secret place] because it is next to a blue number. If 2 turns red the secret place can be either 2 or 3.
>
> (Isabel clicks on number 2; it turns red)
>
> TEACHER: And now? What is useful so as to use as few clicks as possible?
>
> GABRIELA: To click on 1 or 4. If 4 turns blue, the secret place is 2. If the 4 turns red, the secret place is 3.

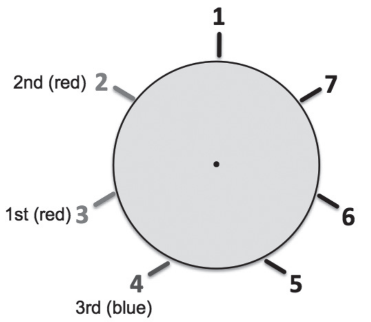


Figure 14

> (Gabriela clicks on the number 4; it turns blue)
>
> ISABEL AND GABRIELA: The secret place is the 2.
>
> GABRIELA: Because it is next to 1... and as the three red numbers must side-by-side, the number 1 must be red and 7, 6, 4 and 5 must be blue numbers. Therefore, as 2 is between two red numbers … it is the secret place.

The trajectory of reasoning of this second pair of students seems to depend on what detail is focused on first. This pair paid attention to the information revealed by a blue, whereas the first pair concentrated on the reds. Each time they got a blue they excluded the adjacent places from the possible positions of the secret place.

Isabel used a strategy of clicking a place next but one to a place whose colour is known to be blue, or perhaps, not clicking a place adjacent to one that is blue. The two strategies are not identical, since one says where to click and the other says where not to click. Even though she announced the second she may easily have been enacting the first.

Magic Square Reasoning

Setting and method

The Magic Square Reasoning was used by the teacher Cláudia with two different grade 7 classes (aged 12) in a lesson lasting 90 minutes. One researcher (one of the authors) was present, taking notes, videotaping the lesson and supporting the students work by clarifying some aspects of the tasks during their activity. The written sheets of all students were collected and analysed. Students' names are pseudonyms.

Lesson descriptions

The task was proposed to students much as here. In the first class, the teacher read the task with the students, highlighting the properties of a magic square (equal sum of columns, lines and diagonals). She asked them to draw magic squares on the sheet she gave them. No other suggestions or questions were made in the first phase of the lesson. The students worked mainly in pairs.

Students appeared to be unclear about what they were supposed to do. The teacher in the classroom explained to the pairs that they should try to find blue and red cells for which the respective sums are the same. The students started to draw some pictures.

The first class

Students started to draw squares with the same number of blue and red cells (or yellow and green), as shown in the work of Rui (figure 15).

Figures were crossed out because they were deemed not to display the required relationship. The fact that colours different from those in the task were used suggests that the student had discerned the detail of cell colour not as a significant attribute in itself, but as an indicator of a relationship. He was aware that the specific colours were not relevant, but rather that the colours distinguished two sets of squares. In other words, he was attending to some relationship between the different colours not the colours themselves.

What seems to be invariant is an equal number of cells of each colour, suggesting that this was the relationship he picked up from the task setting, and so this is what he used to draw his own diagrams. It is not clear whether he had any sense of numbers in the cells or the sum of the numbers in the cells of a given colour. Perhaps he was assuming that the red cells were equal to the blue ones, either individually, or perhaps as a sum, rather

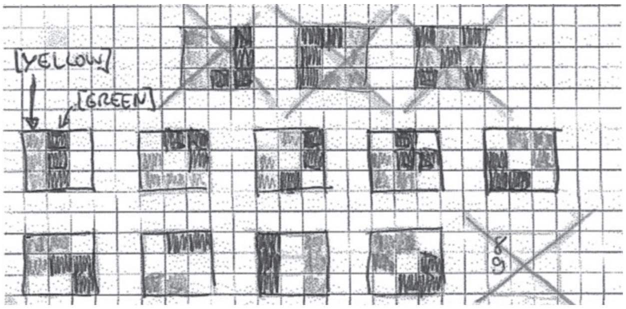


Figure 15.—Rui's drawings

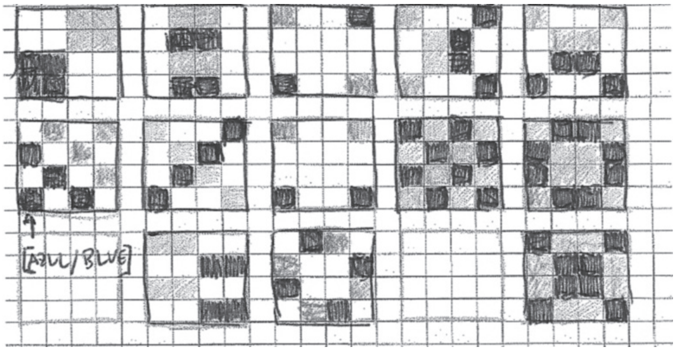


Figure 16.—Rute's drawings

than being something to deduce; perhaps he ignored this aspect, at least at first. In the right hand bottom corner there are some numerals in cells suggesting some sort of a link to cell values.

Another student drew diagrams for a 4 by 4 square which also suggest that her attention was centred in instantiations of the property of having the same number of squares of each colour, augmented by some sort of patterned relationship between the cells of the two colours (figure 16).

At least one student appeared to have misinterpreted the task. The fact that the sum of any two red cells equals the sum of any two blue cells seems to have been taken as a fact, so the task turns into one about patterns and perhaps symmetry (figure 17).

The multi-coloured second drawing and his commentary uses colours as signifiers for unknown numbers. The reasoning is beautiful, but starts from an inappropriate assumption about a property (any two reds and any two blues have the same sum) which is then instantiated in the several colours. From a teacher's perspective, he is frustratingly close to employing precisely the kind of reasoning intended by the task's author.

After talking with the teacher about this, Nuno appreciates that it is not possible to conclude that the drawing displays a valid relationship, and finally he writes: "This pos-

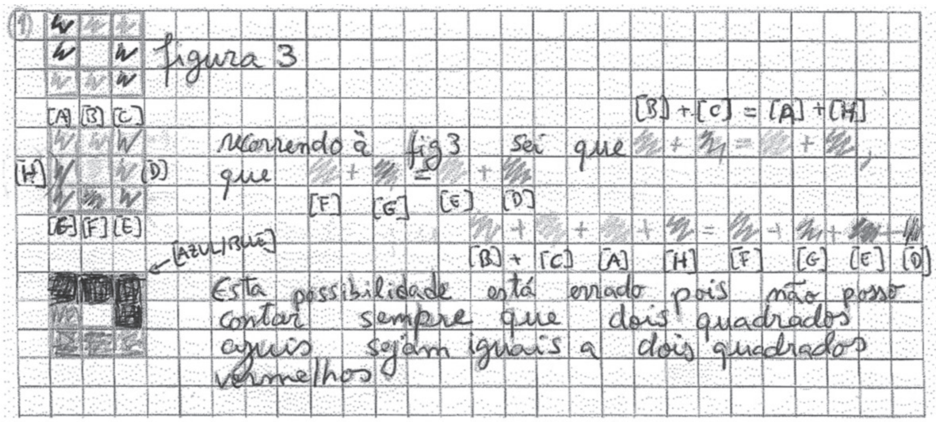


Figure 17.—Nuno's drawing and reasoning

sibility is wrong because I cannot conclude that always the [sum of] two blue squares equals the [sum of] two red squares".

As is so common in human interactions, and particularly in classrooms, students make sense of what they hear, in their own terms. In other words, there are several tasks: the task as imagined by the author, the task as intended by the teacher, the task as interpreted by the students, and the task as actually enacted by the students. Here they discerned individual cells and the fact that they could colour some of them in one colour and some of them in another. However they appeared to be unaware of an imposed property that characterises magic squares (row, column and diagonal sums being equal). The students acted within what they were personally stressing (and consequently ignoring).

They start to attend to the geometric patterns formed by the blue and red cells, drawing several squares with the same amount of red and blue cells. They overlooked the conditions that the numbers in the magic square must fulfil. They may, if they thought about cell values and sums, have considered that the sum of the same number of blue and red cells would always be equal, but they may not have attended to this property.

The second class

From this first experience with the task, the teacher and the researcher understood that it was necessary to emphasize the properties that characterise a magic square. They conjectured that maybe it would help students to understand the involved relationship if they started by analysing the second picture in the task and inviting students to think about why the relation holds in that case. Later with another grade 7 class, the teacher started by discussing with them what a magic square is and asking students to concentrate on the claim about the equality in the second figure and the reasons for its validity. After that she asked for other examples.

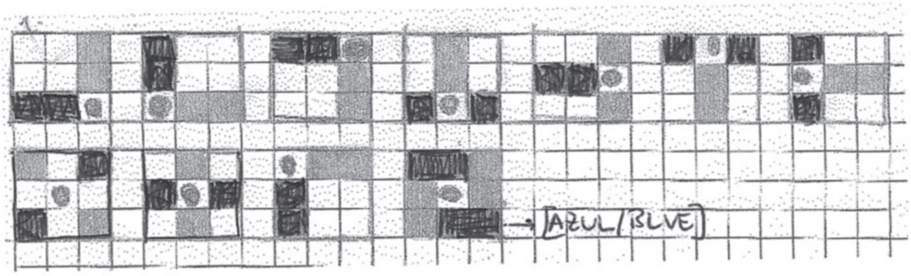


Figure 18.—André's drawings

Many students noticed that the red and blue cells in figure 2 come from lines of cells with one cell in common. They drew several examples of patterns with the same characteristic, marking the common cell in some way. André's drawings are typical (figure 18).

This student also marks a cell that is common to two lines of cells. The last figure, which is figure 3 of the task, could be seen as three pairs of lines of cells all with a common element, or a copy of the third figure of the task, or the combination of the previous two drawings with the colours reversed. A circulating teacher needs to be careful about what they assume the diagram means for the student.

What are the students actually attending to? A teacher dwelling in the mathematics (the constant sum property) so that the sum of cells with one colour must be equal to the sum of cells of the other might overlook or over interpret what students are attending to. Here it could be that the relationship 'lines of cells sharing a common cell' is in the foreground of attention.

Some students drew examples like André but also specialized by inserting some numbers (figure 19).

Interestingly, the third figure in the first row is not a valid design, and it seems as though there has been an attempt by the student to check it using numbers. Although Sofia uses numbers, presumably to check the equality of sums property, she does not use a magic-square configuration. Might she be instantiating the property but unaware of, or forgetting that the numbers are supposed to come from a magic square (figure 20)?

Sofia explains:

> I obtained this information by changing the colours and the positions of each square (cell). With this information I obtain that the [first] green number [1] is common both to the line and the column.

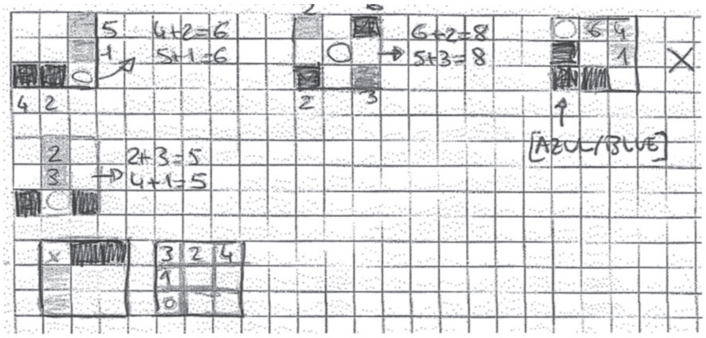


Figure 19.—Manuel's drawings with numbers

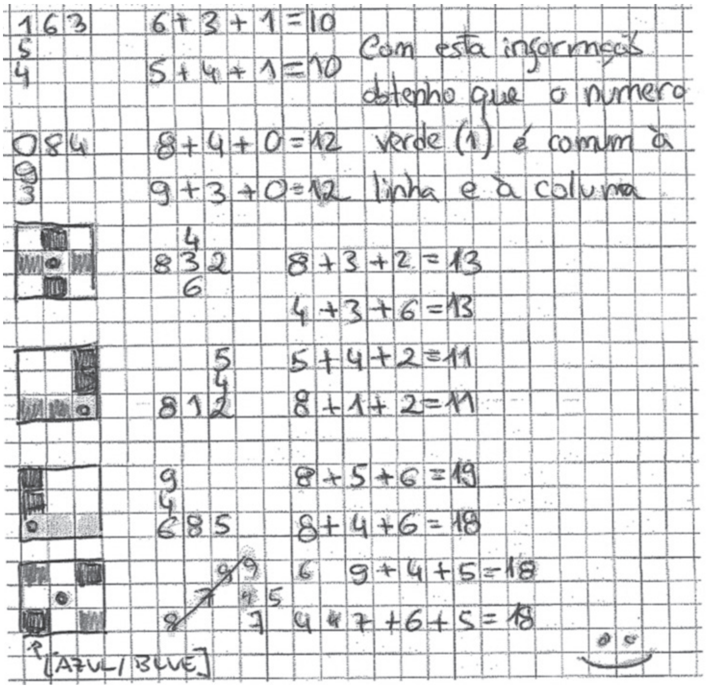


Figure 20.—Sofia's drawings

What she says suggests that she is starting from the sum of the red and the blue cells being equal, and confirming that including the common cell preserves the equality. The task is offering opportunity to use the pr operty of preserving equations by subtracting the same thing from both sides (eliminating what is common) in order to deduce relationships. The relationship between what she is doing and what the task designer intended (one is a 'doing' and the other an 'undoing') is available for discussion. The fact that the green (common) numbers can be any number she likes (ignoring the magic square

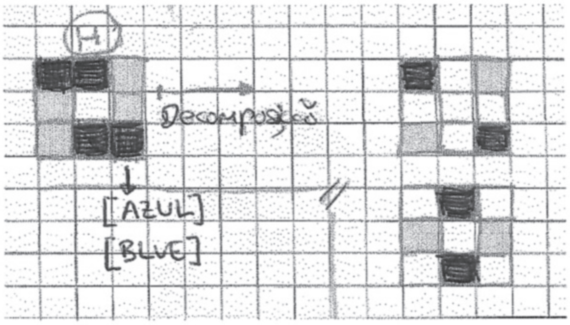


Figure 21.—Tiago's drawings

for a moment) is available for discussion so that she experiences the generality of what she is doing.

What can we make of what the students are doing with the numbers? Are they confirming to themselves that if the cell lines have the same sum then so do the shaded squares with the common cell missing or are they confirming that if the coloured cell sums are equal then so are the corresponding lines of cells? A relationship has been recognized and turned into a property that is instantiated, but the relationship does not make use of the background magic square structure. So the particular instances of the numbers reveals attention focused on the algebraic property that can also be described as 'you can subtract the same thing from both sides of an equation and still preserve the equality'.

Some students look at the third figure in the task as resulting from combining two simpler figures already identified as 'true' (the ones that have one common cell between the groups of red and blue cells). (See figure 21)

This is the sort of reasoning that was intended, though whether Tiago appreciates what the diagram says mathematically in terms of sums is open to question. The same happens with some of the work the students do with the 4 by 4 squares (figure 22).

Pedro claims that "the figure 4 is true" because in a magic square the sums of the cells in each diagonal are equal. From there he notices that by decomposition of that square he can obtain figures 2 and 5, and considers that these two are also 'true'. Since the figures were probably drawn in sequence, the second figure suggests that attention has shifted back to 'two reds and two blues' as acceptable configurations, or else the configuration of diagonals in the 3 by 3 case has been extended to the 4 by 4 case.

This provides a clear example of how reasoning on the basis of agreed properties depends on being clear about the 'agreed properties'. Although the diagonal sums are indeed equal, there seems to be some underlying assumption that pairs of coloured squares are also equal.

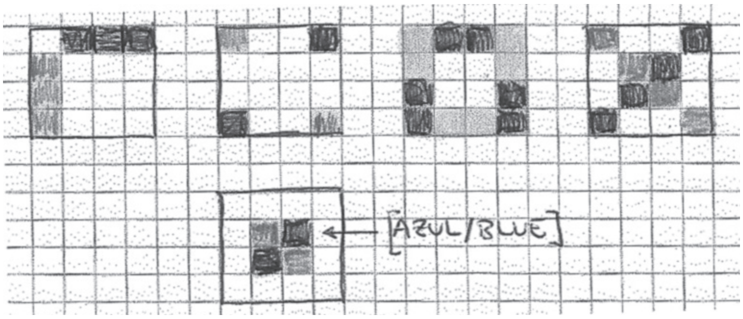


Figure 22.—Pedro's drawings with 4 by 4 squares

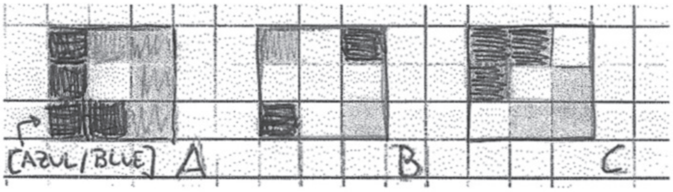


Figure 23.—João's drawings with 3 by 3 squares

João presented a conjecture to the whole class concerning figure C: "the figure will hold 'true' only if the all squares were equal" (figure 23).

However Ruben disagreed and, after working for while on his own, explained why:

> My colleague João presented this conjecture [see figure in drawing below] and said that to make this conjecture [true] it would be necessary to repeat numbers and I tried to find a way of using different numbers in the magic square in order for the sum [of the blue and red cells] to add to the same number . But I did not succeed. Because 9, 4 and 3 adds to 16 but 1, 6 and 7 adds to 14. Therefore it is not equal, so this conjecture is very difficult to obtain in a magic square" (figure 24).

Ruben starts with an actual magic square but then tries to make modifications to it, perhaps seeking an example where the three red and the three blue do indeed have the same sum. The lovely conclusion "this conjecture is very difficult to obtain in a magic square" leaves open the possibility that it can be done, without stressing that it certainly does not always hold. It is a pity to have to make use of a particular magic square, because the idea of the task is to avoid using empirical reasoning or example-checking so as to maintain attention on the process of reasoning on the basis of agreed properties. Thus, looking at the diagram eliminating the two pairs of coloured cells that we do know are equal, one red and one blue cell remain, and these can only be equal if the cells contain the same number (which never happens in a proper magic square).

Ruben articulated the intended reasoning as follows:

Figure 24.—Ruben's examples

> In a magic square the sum of each line, column or diagonal as to be the same. In this case if the white cell is common to both, these (points to the blue cells) will be equivalent to the same so that when added to these (points to the red cells) they will result in the same number."

After having his attention drawn again to the second figure of the task, Nuno also articulates the reasoning behind it:

> the sums of the two [pairs of] cells have to be equal, because if they have one common element, the same value has to be distributed among the two other squares [cells] in order the sum to be the same. However the values in each square [cell] may be different as long as the sum [of both cells] is the number x that added with the common value [to the other two cells] is equal to the desired number in the magic square.

Nuno, like other students, applied this kind of reasoning to the 4x4 magic square. But he questions himself whether the last figure he draws is "true" (figure 25).

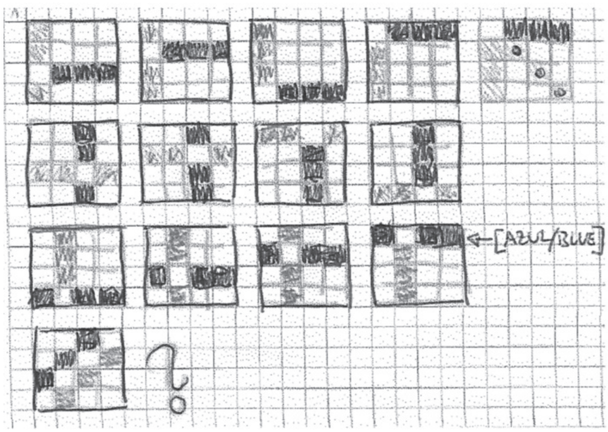


Figure 25.—Nuno's 4 by 4 drawings

Attention seems to be focused on what different row and column configurations might look like where the sums are equal. The fifth one with the diagonal in a third colour might be presenting the fact that row, column and main diagonal sums are all equal. This is a good example of how being rather more systematic in exploring the row and columns of the 4 by 4 square, the student is drawn to question other possibilities.

Overall reflections

We investigated in this paper one conjecture concerning whether the notion of forms or structures of attention can shed light on what students are doing on tasks that call for justification. There are many instances, in Secret Places, of spontaneous use of reasoning by contradiction as well as deduction, showing that young children have access to this sort of reasoning, and raising the question of how teachers might be able to draw upon these undoubted powers in their teaching. However there are several rivers to cross, for example between reasoning in a particular situation and formulating reasoning that works in general, between having to confirm a deduction in the particular and not needing to confirm it, and between finding the secret place efficiently and developing an algorithm that always works no matter what sequence of reds and blues are revealed.
In this task, the natural prolepsis seen is likely to be a factor in clarifying the reasoning and in developing an algorithm for finding the secret place in the fewest clicks. The shift in perception, in what is attended to, that is signalled by indefinite pronouns and by 'missing nouns' may lie at the heart of difficulties in reasoning, rather than the actual reasoning itself being difficult.

The children tackling Secret Places displayed confident and effective use of both deduction and reasoning by way of contradiction. They quickly assimilated the 'axioms' or properties they had available for reasoning. Sometimes, especially at first, they concentrated on one place at the table, and the consequences of its colour, without bringing to bear other information they had. Each of the two pairs nevertheless began to take into account multiple foci consisting of information from the colours of more than one place. They all quickly adopted a property-perceiving stance, though for some this developed during initial play with the applet.

The students tackling Magic Square Reasoning similarly displayed effective use of deduction but they did not have a context for using contradiction (due to an intentional absence of specific magic squares). It was sometimes less clear than with Secret Places whether attention was dwelling in recognising relationships in the particular, or perceiving properties as being instantiated. The first introduction of the task Magic Squares, in which students back-grounded or ignored the defining properties of a magic square, led to a modification of the task presentation. In the second presentation, attention was directed explicitly to the defining properties of a magic square. Reflecting on what happened, it is clear that students were able to engage in reasoning, however, reasoning on the basis of agreed properties depends on being explicit about those agreed properties.

Whereas some of the students earlier in the report focused their attention in unintended ways so that the properties they were working with were not those intended (e.g. diagrams must have equal numbers of the two colours), students who appreciated the defining properties of a magic square were able to reason effectively using those properties as axioms.

As a teacher trying to foster and sustain mathematical reasoning, it is all too easy to misinterpret what students intend with a diagram, so listening to what students say in justification is important. But there is more. If teachers *listen for* what they expect, they are likely to misinterpret what they hear as what they are expecting, or else to classify as incorrect perfectly valid reasoning based on an unintended property. Brent Davis (1996) distinguishes *listening to* from *listening for*, and one way to work on this is to try to be sensitive to indications of what details students are discerning, what relationships they are recognising, and what properties they are perceiving as being instantiated.

Experience with this task showed that students' can reason mathematically. However, even though their reasoning can take a form which is valid in general, it can be founded on inappropriate or unintended assumptions, including a particular instance. The astute teacher pays close attention to the unspoken assumptions being made, and is on the lookout for evidence that students have appreciated the available scope of generality. This applies not simply to this task but to any and all mathematical lessons.

One way to pose the Magic Squares task so as to stress the defining property of a magic square would be to provide one or more specific instances of magic squares as test examples. However the intention of the task was to invoke reasoning without recourse to specific numbers, thereby activating algebraic thinking without identifying it as such. In workshops in the UK with teachers, Mason usually starts with a very brief glimpse of a particular and familiar 3 by 3 magic square, then fades out the numbers, in order to emphasise that the specific numbers are of no interest.

By interpreting students utterances as evidence about both what they were focusing on and the structure of their attention, it was possible to gain insight into children's natural powers of reasoning, and to suggest ways of working with children in order to develop those powers so that they could reason reasonably in mathematics in the future.

Acknowledgements

The authors are grateful to the teachers, Cláudia Torres and Manuel Pereira, and to their students who worked on the tasks and provided us with rich data.

Notes

1 It is too much to expect young children to reason formally, but it is, we claim, well within their possibility to reason 'reasonably'; we like the subtle interaction between 'reasoning' and 'reasonably'.

2 In the classroom, cells were colored red and blue rather than sahaded and speckled

3 The students' names are pseudonyms.

4 Available from `http://mcs.open.ac.uk/jhm3/Applets%20&%20Animations/Applets%20&%20Animations.html` (item 12).

5 The image on the computer screen is colored. For instance, the box concerning "New Mystery Place" is yellow as well as the inside of the circumference. The box labeled "Hot Marker" is red and the "Cold Marker" is blue.

References

Burger, W. & Shaunessy, J. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, *17* (1), 31–48.

Clements, D. & Battista, M. (1991). Van Hiele levels of learning geometry. In F. Furinghetti (Ed.), *Proceedings of PME XV* (Vol I, pp. 223–230). Assisi.

Davis, B. (1996). *Teaching mathematics: towards a sound alternative*. New York: Ablex.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht: Kluwer.

Fuys, D., Geddes, D. & Tischler, R. (Eds.) (1985). *English translation of selected writings of Dina van Hiele-Geldof and Pierre van Hiele*. Brooklyn: Brooklyn College School of Education. (ERIC document 289 697)

Kline, M. (1980). *Mathematics: the loss of certainty*. New York: Oxford University Press.

Mason, J. (2003). Structure of attention in the learning of mathematics. In J. Novotná (Ed.), P*roceedings of the International Symposium on Elementary Mathematics Teaching* (pp. 9–16). Prague: Charles University.

Mason, J. (2010). Attention and intention in learning about teaching through teaching. In R. Leikin & R. Zazkis (Eds.), *Learning through teaching mathematics: development of teachers' knowledge and expertise in practice* (pp. 23–47). New York: Springer.

Mason, J. (2010a). Revealing Shapes: applet freely available for use or download at `mcs.open.ac.uk/jhm3/Applets%20&%20Animations/Applets%20&%20Animations.html` (accessed May 2012)

Mason, J., Burton, L. & Stacey, K. (1982/2010). *Thinking Mathematically* (Second Extended Edition). Harlow: Prentice Hall (Pearson).

O'Brien, T. (2006). What is Fifth Grade? *Phi Beta Kappan*, January, 373–376.

Poincaré, H. (1956 reprinted 1960). Mathematical creation: lecture to the Psychology Society of Paris. In J. Newman (Org.), *The World of Mathematics* (pp. 2041–2050). London: George Allen & Unwin.

Swift, J. (1726). (Ed. Herbert Davis). *Gulliver's Travels*. (Vol XI, p. 267). Oxford: Blackwell.

van Hiele, P. (1986). *Structure and Insight: a theory of mathematics education*. Developmental Psychology Series. London, UK: Academic Press.

Abstract. Two tasks designed to encourage mathematical reasoning without any need for calculations were presented to students with the aim of seeking evidence of different forms of attention (Mason, 2003), and using these to learn about the tasks and about students' power to reason 'reasonably' in mathematics. The first task that involves locating a secret place using an applet was solved by two pairs of grade 4 Portuguese students. The second task, involving the structure of magic squares, was proposed to two classes of Portuguese students, aged 12-13. The interactions between pairs of students and teacher probes were taped and transcribed, and, in the second one, students' written responses were collected as well. In both cases, the data were analysed using the fivefold framework of microstructure of attention. The first case gives evidence that young students can reason 'reasonably' but that there are delicate shifts which may require sensitivity on the part of the teacher to help students progress. In the second case, the data analysis shows that these students displayed the power to reason 'reasonably' in mathematics, and that difficulties can be accounted for in terms of not only what was being attended to, but the form of that attention.

Keywords: Mathematical reasoning; structures of attention; mathematical tasks; basic education

Resumo. Foram propostas duas tarefas a alunos portugueses do ensino básico para os encorajar a raciocinar matematicamente sem o recurso a cálculos, com o objetivo de procurar evidências sobre diferentes formas de atenção (Mason, 2003) e de usar essas evidências para refletir sobre as tarefas e sobre a capacidade dos alunos raciocinarem "com razoabilidade" em matemática. A primeira tarefa, centrada na localização de um lugar secreto usando uma applet, foi resolvida por dois pares de alunos do 4.º ano de escolaridade. A segunda, envolvendo a estrutura de quadrados mágicos, foi proposta a duas turmas do 7.º ano de escolaridade. Em qualquer dos casos, procedeu-se à gravação e transcrição das interações entre alunos e professor e na segunda tarefa recolheram-se, ainda, as resoluções escritas dos alunos. Os dados obtidos foram analisados usando um modelo composto por cinco formas ou microestruturas de atenção. No caso da primeira tarefa, evidencia-se que as crianças são capazes de raciocinar "com razoabilidade" mas que o seu progresso depende de mudanças subtis que requerem sensibilidade por parte do professor. A análise de dados relativos à segunda tarefa, revela, igualmente, que os alunos mostram poder para raciocinar "com razoabilidade". Revela, ainda, que as suas dificuldades podem ser explicadas não só em termos daquilo que é o objeto da atenção mas também da forma desta atenção.

Palavras chave: Raciocínio matemático; estruturas da atenção; tarefas matemáticas; ensino básico.

■ ■ ■

JOHN MASON

University of Oxford & Open University
j.h.mason@open.ac.uk

HÉLIA OLIVEIRA

Instituto de Educação da Universidade de Lisboa
hmoliveira@ie.ul.pt

ANA MARIA BOAVIDA

Escola Superior de Educação do Instituto Politécnico de Setúbal
ana.boavida@ese.ips.pt

(Recebido em maio de 2012, aceite para publicação em outubro de 2012)