# Connecting Real-World and In-School Problem-Solving Experiences

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In this article, we address the challenge of supporting students in *real-world* problem solving as a vibrant part of their *in-school* experience of mathematics. We view this as a two-sided design problem. On the one hand, we argue that authentic problem solving experiences should capture important aspects of the nature of mathematical problem solving as it occurs in the *real world*, *outside of school*. Problem solving of this kind must go beyond simple applications of isolated concepts and procedures learned in a single chapter of a textbook. On the other hand, we argue that problem-solving experiences should serve to motivate reflection on core disciplinary ideas, representations, and techniques. Learners need curricular support to unpack, make sense of, and extend the insights and intuitions that have allowed them to solve problems. Thus, we advocate activities that extend problem-solving session, taking students beyond particular real-world problems to explore the generality and mathematical significance of their solutions. Moreover, the problem-solving experience itself should serve to initiate lines of inquiry in the classroom that develop the idiosyncratic ideas and questions that have arisen in problem solving toward more thorough, formal, and standard *shared* understandings in the classroom. We recognize that it is a significant challenge to fulfill both of these requirements. However, learning environments that achieve this goal can offer powerful opportunities for learning and research, provoking and illuminating two key facets of the learning process: the construction and formalization of knowledge structures.

To address the *real-world* side of our proposition, we consider powerful aspects of mathematics and problem solving as they are experienced in authentic contexts outside of school. We show that the gap between this *live* and *authentic* practice of mathematical problem solving and the typical experience of school mathematics is wide and, in fact, growing wider with changes in the world of work over the last thirty years (Darling--Hammond, Herman, & Pelligrino, 2013; National Governor's Association Center for

Best Practices and Council of Chief State School Officers, 2010; Wolfram, 2010, 17:15). We outline features of real-world problem solving and aspects of the mathematics used in such situations, as a way to indicate the magnitude of the challenge we are facing. Then, we describe the design and rationale of Model-Eliciting Activities (MEAs), a type of learning environment designed to engage learners in simulations of real-world problem solving of the kind described above.

To address the *in-school* side of our proposition, we show how MEAs can be embedded in flexible curricular sequences, called Model Development Sequences (MDSs), which offer learners opportunities to reflect on, unpack, refine, and extend the ideas that they have developed in MEAs — opportunities too often neglected in real-world problem-solving settings (Lesh et al., 2003; Zawojewski et al., 2013). We review research on MDSs and we identify a need for further work to refine design and implementation principles to guide their use. In particular, recent research has sought to understand the need for MDS activities in particular modeling situations (Brady, Lesh, & Sevis, 2015), and to study the effects of engaging in course-length implementations with them (Lesh, Carmona, & Moore, 2009).

As a contribution to the work on elaborating principles behind the MDS construct as a means to extend problem-solving as a core element in course-long engagements in modeling, in the second half of this article we present a detailed episode from an ongoing design study. This episode involves work by the first author with in-service educators in a Masters program in Mathematics Education at a public South American university. We describe instructional decisions he made around a single MEA and how he responsively assembled an MDS sequence for the course through a process of reflection-in-practice (Schön, 1983). Moreover, we show how this process of elaborating a particular MDS illuminated design principles for structuring MDSs in general. We present a set of six principles that resulted from discussing this episode among the author group and we identify resonances between this work and intuitions that we have formed over a long period of working with MDSs. Finally, we close by arguing that these curricular sequences can offer a powerful combination of real-world problem solving experience on the one hand, and rich opportunities for students to reflect on and formalize knowledge on the other.

## Motivation: Mathematical Problem-Solving Outside of School

In recent decades, the experience of mathematical problem solving *outside* of school has changed in fundamental ways (Lesh, Hamilton, & Kaput, 2007). These shifts, we believe, can and ought to be reflected in the kinds of mathematical problems and modeling activities that we present to learners *within* the school curriculum. Unfortunately, the typical experience of mathematics problems in school is out of step with these realities. In this section we describe this mismatch, focusing on two of its facets — the mathematical *tools* involved, and the problem-solving *practices* employed. Finally, we introduce a design approach to address these issues.

## Mathematical Tools for Real-World Problem Solving

The contemporary world of work is experiencing rapid changes, which affect the mathematical tools needed to model its most important problems. In particular, the following trends have had significant impact: on one hand (a) an increasingly pervasive understanding of the world in terms of large data sets whose patterns indicate connectivity, feedback, emergence and complex systems as opposed to simple input-output relations, and on the other hand, (b) lowered barriers for students to participate in creating, exchanging, and critiquing *computational* representations of these complex systems and emergent phenomena. Together, these factors are associated with increasingly dominant disciplinary theories or *paradigms* (Kuhn, 1970) for thinking about and explaining causality, across disciplines. We argue that these features present fundamental challenges and opportunities to the enterprise of school mathematics in fulfilling its role to prepare learners to participate effectively in discourses about causal relations.

In general, our experience of the world is characterized by an ever-more rapid flow of information, a high degree of connectivity, and extreme interdependence. Computation is pervasive and mediates near-instantaneous interactions among the 'nodes' of a connected global network. Popular texts (*e.g.*, Barbasí, 2002; Watts 1999; 2004) have made the general public aware of an accelerating trend in mathematics and the sciences: the tendency to view core scientific phenomena as the emergent effects of the operation of *complex systems* (c.f., Wilensky, Brady, & Horn, 2014). Although certainly such approaches had previously been used by researchers in the mathematical, physical, chemical, and even the biological sciences (*e.g.*, Motter & Albert, 2012; Piela 2014; Stamatakis, 2006), a fundamental change has occurred in that complex systems are also seen to characterize the relations among "players" in *human* and *social* systems (*e.g.*, Axelrod, 1997; Epstein, 1999, 2012, 2014; Gilbert & Troiztsch, 2005; Miller & Page, 2007). Along with the rise of the Internet (and often using the Web as a key example), the relations among individuals, countries, companies, and other institutions are increasingly framed in terms of networks of relations. These networks express the structure of relations of communication, influence, and causality; and they often exhibit feedback loops, redundancy, and many-to-many relations. All of these features require STEM professionals to reconceptualize tools that they have learned in traditional school instruction to apply them flexibly and creatively.

For instance, in the mathematics of change, capturing the essence of relationships of causality and co-variation, the notion of *function* is central. However, the modeling toolkit developed in most students' experience of high-school mathematics consists mainly of *analytic* functions, and in particular, combinations of polynomial, trigonometric, and exponential/logarithmic functions. Furthermore, these functions are generally studied in situations of "simple" causation — that is, in situations where a single input produces a single output, with no systemic feedback. In the complex, connected world we have described, functions serve as useful *building blocks* for modeling aspects of systems, within specific regimes, but they must be conceptualized in a larger and more flexible context to describe systems as a whole.

Fortunately, our second point about the world of today offers a means to enable students to make the shift to creative and flexible use of mathematical tools. For the study of change in complex interdependent relationships, the traditional apparatus includes systems of differential equations. However, computational systems modeling environments such as STELLA and VenSim (*e.g.,* Roberts et al.*,* 1997; Shii & Gill, 2005), and agent-based modeling environments such as NetLogo (Wilensky, 1999), StarLogo TNG (*e.g.*, Klopfer et al.*,* 2009), and Repast (Collier, 2003) allow learners to approach such systems without the advanced apparatus of differential equations.

In general, technical and other barriers to participating in the discourse on complex causation are lowering. This certainly does not guarantee equitable access to powerful tools, however, and this is a core challenge for schools to address. Nevertheless, barriers to accessing information and even to participating actively in creating and publicly sharing computational models of complexity are rapidly disappearing. In addition to the modeling environments above, key tools such as spreadsheets and graphing tools of various kinds have now been expressed in free and open-source software. The price of computational hardware has likewise continued to decrease. Thus, it is increasingly feasible to make meaningful changes in traditional educational approaches across a diverse range of settings and contexts, even when these innovations depend on access to computational tools. This state of affairs identifies an opportunity and an obligation for us to envision a school mathematics that is responsive to the need-to-participate in the discourse about emergence and complex causation.

Most of all, today's mathematical problem solving requires that students have the capacity to use functions and other mathematical tools as means for *conceptualizing* the world as framed by the problem. In authentic problem settings, mathematical forms are not simply dressed up in thin disguises to be discovered easily by learners. Instead, when problem solvers apply mathematical tools to solve such problems, they engage in a radically *interpretive* process. They tentatively impose mathematical structures on the world and iteratively refine these structures to improve the viability of the match. This process both requires and fosters a high level of fluency with the constituent mathematical tools. It raises rich and fundamental questions both about the nature of modeling and mathematical interpretation and about the deeper functioning of the mathematical tools and structures themselves.

## Mathematical Practices for Real World Problem-Solving

Alongside the mismatch between the mathematical *tools* needed for modeling in authentic settings and the experience of these tools in school, we also see fundamental mismatches in the *practices* used to attack problems in and out of school. In particular, we identify below five basic features of authentic problem settings as experienced by professionals in mathematically rich domains beyond school. There is an increasing recognition that these features of problem-solving practice are *fundamental*, in that they have a significant impact on the kinds of thinking that problem-solvers engage in.

**Working in Groups.** While not *all* important problem-solving work in the real world is done in collaborative groups, most problems that pose significant challenges to an organization do require solutions that tap into varied areas of expertise. So, if the problem goes beyond a simple application or minor adaptation of an organization's prior experience, a team will generally be assembled to conceptualize or execute the work.

**Focusing on the Needs of a Definite Client.** Realistic problems almost always are situated with respect to a particular client who has a definite need. The client's context fundamentally shapes not only the criteria by which solutions are judged, but it also conditions a myriad of operational decisions. Depending on the ways in which the solution will be *used* by this client, problem solvers must be sensitive to which kinds of errors might be acceptable or unacceptable; what degree of precision is needed or appropriate; what kinds of input measurements or calibrating measurements are available; and so forth. Rather than merely offering adjustments or adaptations to a context-free solution, these factors can often fundamentally shape the solution and the problem solving process.

**Addressing Challenges in Measurement and in Operationalizing Constructs.** In realistic situations, knowledge of the problem situation may be incomplete or based on data sources that must first be interpreted in order to be applied to the problem. Even more importantly, qualities that the client wishes to optimize may be inadequately or incompletely specified or operationalized. For example, in forming a team of salespeople for a new assignment, a client may wish to optimize revenue production. However, even if the salespeople are to be drawn from a pool of existing employees for whom reliable historical data is available, there is a challenge in creating an operational construct of "salesperson productivity" and using data from historical contexts to predict this value of productivity in the new context. Different salespeople are likely to have performed better or worse in different conditions, and many will have demonstrated positive or negative trends in their performance over time. Different operational definitions of "productivity" may lead to significant changes in the ranking of candidates and thus in significant changes in the proposal to the client.

**Providing Solutions that Address Uncertain or Changing Conditions.** In many authentic problem-solving settings, groups identify key conditions or parameters that may trigger radical changes in the solutions required. Moreover, in some cases, the values of these conditions or parameters may be unknown or may even vary within the client's decision-making context. Thus, in addition to coordinating a variety of mathematical tools and structures in their solution, teams may need to specify conditional logic for applying one solution or another. For instance, they may recommend that "under the following conditions or assumptions, apply procedure #1, but under contrary conditions, apply procedure #2." In particular, such ideas go beyond specifying the domain of solution functions: the thinking involved is significantly more sophisticated, though clearly related.

**Being Responsive to Key Trade-Offs.** Although this feature is related to conditional thinking, it emphasizes criteria by which different viable solutions will be compared or

evaluated. It is common in engineering contexts that solution strategies are judged not only for the functional qualities of an end product but also according to how they balance "non-functional" trade-offs between, for example, cost and time (for a product), or complexity and precision (for a process). As with the other features described above, we argue that these considerations are intrinsic to the modeling and problem-solving process. They are not concerns that can simply be added to an abstractly formulated solution: rather, they play fundamental and constitutive roles in *forming* and evaluating solutions.

## A Design Approach to Creating Problem-Solving Activities that Bridge these Gaps

It is a significant challenge to create in-school situations that capture core elements of the practice of problem solving outside of school, as described above. Fortunately, a tradition of educational design research has been working on this problem for over thirty years. Researchers adopting a Models and Modeling Perspective (M&MP) in mathematics education (Lesh, 2003a; 2003b; Lesh & Doerr, 2003) have engaged in research to illuminate the nature of knowing and learning in authentic problem-solving characteristic of real-world settings. A key requirement of such settings is that they challenge learners to engage in original mathematical work (i.e., to produce mathematical interpretations and constructions that are *new to them*), rather than being simple applications of mathematics learned from an authoritative source (*e.g.*, their textbook). Iterative design research to create such learning environments has led to the development of a genre of materials and activities, known as Model-Eliciting Activities, or MEAs (Doerr & English, 2006; Lesh et al., 2000; Hjalmarson & Lesh, 2007; Lesh & Doerr, 2003). In MEAs, students are presented with authentic, real-world situations where they repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to generate a structurally significant product — a *model* — comprising a conceptual structure for interpreting and solving the given problem (Lesh & Doerr, 2003). These MEA activities give students the opportunity to create, adapt, and extend scientific and mathematical models in interpreting, explaining, and predicting the behavior of real-world systems.

**An Example MEA: Summer Jobs.** The Summer Jobs problem (Chamberlin, 2005; Lesh & Lehrer, 2000) included as Appendix A, often serves as an initial MEA experience for learners. In particular, it was the first MEA used with the group of South American high school teachers whose work is described later in this article.

In the problem, a table of data is given showing information about a sample of workers in an amusement park during the past summer. The group's challenge is to develop a way to decide which workers should be hired full time, part time or not at all for the following summer. The information given about the past summer includes: (a) the *amount of money* earned by each worker during June, July, and August, and (b) the *amount of time* that each worker worked during periods where park attendance was "slow," "steady," and "active". Thus, solutions to the problem often involve developing some kind of index of *worker productivity*, and taking the attendance conditions of the park into account.

A key outcome of the whole-class discussion of the MEA is the notion that when such a construct is *operationally defined*, a variety of reasonable definitions may be possible. Though none of these are objectively "correct" or "incorrect," all of them involve different assumptions and may lead to markedly different conclusions or judgments. In particular, in real life situations where statistics procedures are useful, results often vary significantly depending on how constructs such as "averages" are operationally defined. For example, in two independent sessions, 98 Ecuadorean high school teachers engaged with this problem in groups of 3–5. Each group produced a solution for the client along with an explanation of their rationale and how their procedure could be applied not only to the particular set of salespeople given but also more generally to any group for whom similar data was available. A total of 27 groups submitted solutions, with judgments shown below:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **María** | C | | P | | P | | | | P | C | C | C | C | | C | | C | | P | C | P | | P | | | P | |
| **Kim** | C | C | C | C | C | C | C | C | C | P | P | C | P | C | | C | P | C | C | P | C | C | C | C | C | C | C |
| **Terry** | | | P | | C | | | | P | | | | | | | | | | P | | | | | | | | |
| **José** | P | P | C | P | C | P | P | P | C | C | | | C | P | | P | P | P | P | C | | P | P | P | P | P | P |
| **Chad** | P | | | | | P | | | | | P | P | P | | | P | | P | | | P | | | C | P | | |
| **Cheri** | C | P | P | C | C | P | P | P | C | P | P | C | C | P | C | P | C | C | C | P | C | C | C | C | P | C | C |
| **Robin** | P | C | | P | P | C | C | C | P | | P | P | | C | | C | P | P | P | | P | P | P | P | C | P | P |
| **Tony** | | C | | | | C | C | | C | | | P | C | | C | C | | | C | P | | C | C | P | C | C | P |
| **Willy** | | P | C | P | P | | P | P | P | | C | P | | P | C | | | C | | C | C | | | | | | C |

Figure 1. Data from the "hiring boards" for Summer Jobs, showing the judgments of 27 groups of teachers at a conference in Ecuador. In the diagram "C" indicates a decision to hire the worker full time (tiempo completo) while "P" represents part time (tiempo parcial).

For both MEA sessions, a "hiring board" was displayed at the front of the classroom and filled in by the groups as they finished formulating their solutions and writing their letters to the client. The visible disagreement in judgments provoked rich and vibrant discussions of groups' strategies. It quickly became clear that differences in results were not due to *errors* in calculations, but rather to mathematically different *operational definitions* of productivity, as well as different procedures for dealing with anomalies in the data (such as workers who did not participate in the business in a given month or who worked significantly more or fewer hours than others). Observing the diversity of solutions and the conflicts in judgments was an important experience for the participants. Although some of the teachers in the group expressed discomfort with the idea that an "optimal strategy" could not be definitively identified, the validity of different approaches to weighing and combining the available data came through in the course of the discussion.

MEAs like the Summer Jobs problem present learners with situations in which familiar mathematical procedures and constructs are *both* applicable *and* insufficient. That is, on the one hand such problems are accessible to learners from a wide range of levels of ability, experiences, or knowledge (from upper elementary school through graduate school). On the other hand, learners encountering these problems find that they have no ready-made "correct" solution process they can apply to address the client's situation.

As a result, learners engage in solution-construction processes that put them off balance in comparison to typical school-mathematics tasks. Moreover, this uncertainty is part of the design of the MEA, helping to illuminate fundamental conceptual issues associated with the core mathematical structures involved.

**Questions that are Frequently Raised about MEAs.** In addition to stimulating vibrant discussions among the teachers about their solutions, the experience describe above also provoked important questions about the nature of MEAs as a type of problem, and how they could be implemented in schools. We respond to three of these basic questions here:

Question 1: How can we give problems to students where there is no single "best" or "optimal" solution? For Summer Jobs and other MEAs, the products that students develop are not simple "answers." In fact, although groups produce a *solution* for the client, the term "solution" may fail to capture the breadth of the conceptual structure that they create. Here as in other MEAs, groups develop *tools* and *procedures*, along with framing *assumptions* and *interpretations*, which are intended to be appropriate and useful to a specific client for a definite purpose. To the extent that the client and her problem are clearly specified, groups are able to *judge for themselves* whether their current response to the problem is adequate, effective, and sufficiently generalizable to be useful. This is absolutely crucial: otherwise, groups see the *instructor* as the audience and judge of their solutions, and the learning situation collapses into yet-another-school-math-problem.

One important note in connection with this question is that it is *not* necessary for the client and her problem to be "completely" specified. In fact, in Summer Jobs, much of the disagreement among operational definitions of worker productivity comes from nuances in groups' judgments about when part-time workers will be used; how predictable are the variations in the park's activity level; and so forth. But it is reasonable that the client herself may not have thought these matters through before posing the problem to her "consultants." This type of incompleteness in the client's specification simply indicates how a model-rich response to a problem illuminates the problem situation itself. After engaging with multiple MEAs, more sophisticated groups of students begin to incorporate multiple scenarios into their solutions, identifying decisions that the client should make or additional information that he or she should collect before choosing among the solutions or procedures that they present.

Question 2: Why are MEAs not more targeted to single topics as found in textbooks or curricula? As we have argued above, one of the main features that distinguish "real life" problems from "typical school" problems is that they usually involve constraints. This is because clients tend to care about partly conflicting factors such as costs and benefits, risks and gains, and so on. And, this means that developing a solution often requires integrating ideas and procedures drawn from more than a single textbook topic area. From an educator's perspective, an advantage of these kinds of problems is that they support learners in making connections between different areas of knowledge and among various "big ideas" of a course. These types of connections reflect how knowledge is held together *in action* — in constructing solutions — as opposed to the types of connections that drive the *logical presentation* of ideas in a syllabus or a textbook.

Question 3: Are MEAs "open-ended" problems? Our answer to this question is "Yes and No." On the one hand, MEAs are open-ended in the sense that they neither have a single "correct" answer, nor do they lead students through the process of constructing a target conceptual system devised by the MEA author. Instead, they provide opportunities for learners to *adapt* and refine their own thinking, leading to a degree of diversity of solutions among different groups in a given class. On the other hand, there are several senses in which MEAs are *not* open-ended problems. First, the needs of the client provide a clear means of judging student responses, so that MEAs are certainly not "anything goes" problems. Second, in most MEAs the problem situation is sufficiently specific that relatively small set of basic mathematical constructs will be useful in formulating responses. Thus, it is fairly certain that the solutions of different groups will be *comparable*, enabling groups to extend their learning by reflecting on each other's work and rationales. Finally, MEAs are extremely compact in terms of the time that they require. Historically, MEAs were limited to a single class period, because researchers wanted to use them to track the development of students' ways of thinking. If an activity took longer than a single class session, researchers would lose the ability to observe students' changes in thinking in the time between sessions, leading to incomplete accounts of the process. When MEAs began to be used for their instructional value, compactness became an asset for a different reason. Many versions of problem-based or project-based learning attempt to provide experiences of real-world problem solving, but most of these formats require weeks of class time or longer. In contrast, MEAs and the extension materials that accompany them can be arranged flexibly by teachers to meet a variety of instructional goals.

## Principles Guiding the Design and Implementation of MEAs

As we have seen, MEAs were originally designed as environments for research into what it means to "understand" important concepts in the K-12 mathematics curriculum, and their goal was to provide documentation and evidence to study the development of ideas in classroom groups. They were designed so that students would clearly recognize the need to develop specific constructs — without dictating *how* they would think about relevant mathematical objects, relationships, operations, patterns, and regularities. Along with a collection of particular MEA activities, the early M&MP community outlined six key design principles for MEAs to meet these goals (see, *e.g.*, Doerr & English, 2006; Lesh et al., 2000; Hjalmarson & Lesh, 2008):

1. *Personal Meaningfulness.* Is the problem situation realistic, in the sense that a solution would be of genuine interest to a client? Is the problem space sufficiently open to ensure that different groups of students are able to pursue diverse solution paths based in their own unique personal knowledge and experiences?

2. *Model Construction.* Does the problem truly require the new construction, modification, adaptation, or extension of a model in order to be solved? Does the problem engage with deep mathematical structures and regularities, rather than engaging mainly at the surface level?

3. *Self-Evaluation.* Are the client's needs and constraints sufficiently clear that student groups can judge for themselves the usefulness or adequacy of proposed solutions?

4. *Model Generalizability.* Do the models that are created in the activity apply only to the specific situation of the problem, or are they likely to be generalizable to a broad range of situations?

5. *Model Documentation.* Will student responses to the problem explicitly reveal their characteristic ways of thinking about the situation? Will they provide clear evidence about the mathematical objects and relations they have engaged with in solving the problem?

6. *Simplest Prototype.* Is the problem situation as simple as it can be, while still meeting the other design principles? Does the experience of the MEA "stick" with students so that they are able to use it as a lens for viewing future problems that feature similar mathematical structures?

The notion of a *model* is fundamental to both research and teaching with MEAs. For our purposes, models are:

> …conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s) — perhaps so that the other system can be manipulated or predicted intelligently (Lesh & Doerr, 2003, p. 10).

That is, models are sense-making units, interpretive systems that human beings construct, use, and re-use to conceptualize the world around them. *Mathematical* models are models of this kind that emphasize structural patterns in the world.

When groups of learners encounter novel problem situations with specifications that demand a model-rich response, they must develop a shared understanding of the nature of the problem situation: a need that provokes the construction of *shared models*. Then, in developing a solution that takes account of the situation and meets the given specifications, the group's models are observed to grow through relatively rapid cycles of development. These cycles are iterative, in that learners express, test, and revise their ideas, adjusting them incrementally to improve the fit and viability of their solutions relative to the client's specifications. These model-rich problems are therefore settings where significant constructive activity occurs *and* where this activity generates an auditable trail of the progress of the group's thinking. These are powerful features of MEAs both for creating educational activities and for conducting research into learning.

## MEAs and the Affordances of Classroom-Based Learning

We have argued that MEAs are a time-efficient way of providing students with experiences that have important connections with mathematical problem-solving as it occurs in the real world outside of school. In presenting this case, we have run the risk of suggesting that real-world problem-solving contexts provide superior learning environments to those which can be achieved in schools. This is not our intention. In fact, we argue that classroom settings are positioned to offer significantly more powerful opportunities for reflection and unpacking of problem-solving than are typically experienced in the world of work. While "post-mortems" on solutions are in fact a part of many real-world problem-solving efforts, we argue that the classroom environment is particularly conducive to a deep unpacking, extending, and reflecting both on problem-solving processes and on the nature and applicability of the basic tools that a team has used. The result is that problem-solving in school can stimulate curiosity and a sense of the need for basic fluency and effectiveness in operating with elements of the mathematical modeling toolkit. This is an extremely valuable contribution, and we argue that it can feed powerfully back into the learner's problem-solving effectiveness.

### Supporting Model Development: Reflecting, Unpacking, and Extending

To capitalize on this opportunity, recent M&MP research has investigated ways in which MEAs can be embedded within larger instructional sequences, called Model Development Sequences, or MDSs (Lesh et al., 2003). MDSs offer classroom groups opportunities to unpack, analyze, and extend the models they have produced in MEAs, as well as to connect their ideas with formal constructs and conventional terminology. This unpacking work helps to ensure the lasting retention of concepts at the level of generality required to apply them flexibly in novel situations. MDS activities also set the stage for the critical connection between conceptual development and procedural knowledge that is required for students to achieve well-rounded mathematical competence.

In particular, a given MDS may include the following, in addition to one or more MEAs:

- **Reflection Tool Activities (RTAs)**, which support students in stepping back from their modeling processes and reviewing this work as critical observers of both individual and group modeling behavior. M&MP research expects that when students interpret situations mathematically, the interpretation systems they engage are not purely logical or analytical in nature. Rather, they also involve attitudes, values, beliefs, dispositions, and metacognitive processes. Moreover, the M&MP does not treat group roles or group functioning as if these were fixed student attributes that determined their behaviors. Instead, students are expected to develop a suite of problem-solving *personae* that they learn to apply purposively as the situation demands. A wide range of reflection tool questionnaires, formats, and guides have been developed to emphasize different dimensions and objectives in students' process-unpacking work (c.f., Hamilton et al., 2007).

- **Product Classification and Toolkit Inventory Activities**, in which students categorize the kinds of thinking involved in their solutions to MEAs. These activities also help students continue the work of abstraction, identifying links among different solutions to MEAs and between these solutions and the "big ideas" of the course. These activities can involve graphically representing knowledge and links between knowledge elements in various ways.

- **Model Extension Activities (MXAs)**, often involving dynamic mathematics software, in which the class extends and formalizes key elements of mathematical thinking that have appeared in student solutions. And, finally,

- **Model Adaptation Activities (MAAs)**, which allow students to generalize ideas and techniques developed in MEAs, applying them to situations calling for similar performances. These activities can also provide smaller-timescale modeling scenarios that exercise concepts students have explored in other components of the MDS. They may be pursued individually or in small groups, depending on the nature of the task and the teacher's instructional or assessment goals.

These activity types were formulated over the course of years of iterative design research involving MEAs. MDS sequences deploying such activities are conceived as highly modular, to be responsive to the needs and intentions of teachers. Importantly for ongoing research into MDSs, this flexibility requires teachers to make consequential instructional decisions, creating opportunities for *multi-tier* design based research (Lesh & Kelly, 2000, Doerr & Lesh, 2003). Within this framework, a first tier involves the primary modeling activity of students engaging with MEAs and MDS materials. In parallel, a second tier, involves teachers as they develop and articulate their conceptions of their students' learning. These conceptions are *teacher-level models* of student thinking, and the adaptation and implementation of MDSs can be treated as a *teacher-level* MEA. Finally, at the researcher level, a third-tier modeling activity involves researchers openly reflecting on their own models — their conceptions of the development of ideas among student groups, and of the teacher's patterns of observation and response to those students, along with the researcher's own perceptions and actions as participant-observers in this context.

## The Need for Principles to Guide the Design and Implementation of MDS Activity Sequences

Thus, while MDS sequences are designed to be flexibly adapted by teachers, the instructional decisions involved are consequential, and a repository of MDS activities is not simply an *a la carte* menu of materials to choose from. Historically, the M&MP research tradition has used principles of design and implementation to maintain a balance between flexible adaptability on the one hand and principled instructional decision-making on the other. In the discussion above, we have presented the six principles for designing and implementing *MEAs*. These principles were developed iteratively through collaborations between researchers and reflective practitioners over several years of im-

plementation work (Lesh et al., 2000; Lesh & Doerr, 2003). In an effort to articulate a parallel set of principles for orchestrating MDSs, we have been engaged in an extended design research effort (c.f., Brady, Lesh, & Sevis, 2015). The remainder of this article describes one implementation episode from this work, along with the provisional set of design principles that emerged from analyzing that episode within the author group.

## An Implementation Episode from Design Research on MDS Sequences

In this section, we present an implementation episode extracted from an ongoing design research project focused on Model Development Sequences. We begin by laying out the methodology of the project and the research questions that are advanced by this episode. Next, we describe the implementation and the in-the-moment decisions made by the first author in developing an MDS responsive to the needs of the class. Finally, we describe the MDS design principles that emerged from reflecting on this episode with the author group as a whole.

### Methodology and Research Questions

The implementation episode described here is extracted from a broader line of work that adopts a design research methodology (Clements, 2007; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Lamberg & Middleton, 2009; Sandoval & Bell, 2004). Through successive cycles of implementation, analysis and redesign, we investigate core questions about learning and the development of ideas on one hand; and we iteratively refine the materials, learning environments, and design principles behind activities that act as the context and stimulus for learning on the other. The curricular materials in question for this work comprise a course-sized collection of modeling activities in the domain of quantification, data modeling, and statistics.

In this setting, our research proceeds at two different scales, and the implementation episode presented here makes contributions to both. At the local scale, we are concerned with questions associated with linear regression and proportional reasoning, connecting these concepts with representational fluency with Cartesian graphs. At this level, we address two questions:

- How can engaging with graphical interpretations of their thinking both destabilize and enrich learners' interpretations of data that are based implicitly on linear models of variation and/or proportional reasoning?

- What sequences of activities can provide opportunities for learners to struggle with linear models in a broader context, identifying the implicit assumptions, strengths, and limitations of linear models?

At a larger scale, we are investigating the following question:
- What are design principles for Model Development Sequences that activate important connections between big ideas in a course on Quantification and Data Modeling?

*The Implementation Context and the Study Population*

In the episode analyzed here, the first author was the instructor for a three-credit module within a Masters of Mathematics Teaching degree program (Maestría en Docencia de las Matemáticas) at a public South American university, offered in a sequence of evening class sessions. The participants in this course were thirty secondary mathematics teachers, twenty-eight of whom were actively employed as classroom instructors during the daytime. The module was a standard component of the university's Master's degree program and was oriented toward mathematical modeling with functions using technological tools. It consisted of twenty-four hours of class sessions offered in daily sessions during a single week of calendar time. In addition to addressing the required topics of the module, the instructor aimed to expose participants to a variety of research-based pedagogical approaches that foregrounded student participation, construction, and modeling. These included not only activities from the M&MP tradition, but also generative activities (Stroup, Ares, Hurford, & Lesh, 2007; Stroup, Ares, & Hurford, 2005; Brady et al*.,* 2013), and *constructionist* activities (Papert 1980, 1991). The instructor also intended the particular materials used in the course to be responsive to the participants' needs and interests. Pursuing an adaptive approach to the module in this compressed timeframe was a demanding experience; it required extensive preparation before the module began as well as significant adjustments and course-corrections during the week of the implementation itself.

Data collected during the course included artifacts produced by the student groups during individual and small-group problem solving activities; student reflections submitted to a Google Site established for the course module; and the instructor's reflective field notes taken after each session. These data were analyzed in order to characterize the instructional decisions that were made during a single segment of the course module, which constituted an MDS sequence.

We argue that this course module was a particularly appropriate context to engage in reflection on design principles for MDS sequences for several reasons. First, it placed the instructor in a role that blended aspects of teaching and research. Because the classroom group consisted of practicing teachers, they were attentive not only to the mathematical content of the activities, but also to pedagogical moves made by the instructor within and between activities. Second, because adaptability and responsiveness to the diversity of learners' ideas and ways of thinking were explicit instructional values for the module, this put additional pressure on the instructor to take a reflective and principled approach to the selection, ordering, and management of activities to exemplify these values. And third, because the module had an obligation to expose teacher-learners to a variety of types of activities and technological supports, it put pressure on the instructor to construct new links between MEAs and activities that did not historically arise from within the M&MP tradition. Thus, it challenged him to reflect on the roles that activities from outside of the M&MP could play in unpacking and extending the ideas that emerged in MEAs.

### *Description of the Implementation*

In this episode, we focus on one activity sequence of the module, which began with a primary MEA, the Bigfoot problem (Lesh & Harel, 2003, and see Appendix B), along with a Reflection Tool (RT) activity (Appendix C). It then included two extended generative activities, to be described below, which served as Model Extension Activities (MXAs) and the Drill & Skills Problem (see Appendix E), which the instructor used as a Model Adaptation Activity (MAA). Thus, the implementation as realized engaged with many of the activity types that have been used in building MDS sequences, and each of these activities was deployed with a specific pedagogical intention, which we will analyze. Our purpose is to describe the instructional decision-making processes that led the instructor to select these activities from a broader set of materials that he had prepared. These were situated decisions, responsive to patterns of thinking that emerged in the moment in the class's work. They were intended to help the class to "unpack" the solutions they produced during the Bigfoot MEA. They thus illuminated implicit design principles for MDS sequences. Through discussions among the author group, we elaborated explicit versions of these principles, which we present at the end of the article.

In the Bigfoot problem, the classroom space is arranged to portray the scene encountered by a group of cub scouts (niños exploradores) as described in Appendix B. Groups of 3–4 participants are given tape measures and work to make mathematical inferences about physical characteristics of an unknown person ("Bigfoot") who has repaired the drinking fountain in the park and who has left large, distinctive footprints in the mud around the drinking fountain.

After the groups completed their solutions to the problem, the class engaged in a whole-class discussion, during which several factors emerged that the group experienced as puzzling or problematic. The majority of the groups arrived at solutions that involved calculating with *ratios* (using "la regla de tres" — the rule of three). The groups constructed these ratios in different ways — using one or more measures from the variety of available measures (footprint length, footprint width, or stride length); using more or fewer data points (*e.g.*, sampling one or more group members; including data from other members of the class; restricting their samples to people with certain profiles — *e.g.*, males and females, or males only[1]; or sampling both short and tall people; sampling tall people only. An interesting wrinkle for the groups' thinking was added by the presence of one of the teachers' six-year old daughter. Some groups saw measurements of her feet as relevant to the problem, while other groups did not: the discussion of this point raised important ideas about the conceptual models underlying their thinking.

As groups shared their work, it came to light that they had arrived at widely different inferences about the height of Bigfoot. (All groups attempted to infer the height, while only some groups attempted other features such as weight). When the instructor suggested making a graphical representation of the different groups' inference methods, the discussion intensified. Some students were surprised to find that their ratio-based approach implied determining a line that passed through the origin. Meanwhile, the groups that used the 6-year old's measurements felt that such a model was particularly problematic.

(Whereas during group work it had seemed acceptable to average data from adults and children in calculating proportions, the graphical representation made different consequences of this decision more salient.) Some groups advocated a linear fit with a non-zero y-intercept; other groups argued that a model that did *not* pass through the origin was problematic: in the extreme, how could someone have a foot size of zero inches, but a non-zero height?

Gradually, the notion emerged that *no* single predictive linear model could be used satisfactorily over the entire domain of possible foot sizes. Although this idea was powerful, the instructor felt that it was only shakily grasped and not fully taken as shared by the class as a whole. To reach this insight, learners had moved from their comfort zone of a ratio calculation (accompanied by an averaging process if they sampled more than one subject) to connect their reasoning with a graphical interpretation involving a function whose domain was in question. This choice of representation also involved a change of conception from using *arithmetic* to *calculate an unknown* to using a *function* to *describe a relation of covariation* between two measurements. Moreover, in the class discussion some students (correctly) pointed out that even if a group produced a close-to-accurate inference about Bigfoot's actual height, this did not necessarily mean that their *modeling strategy* was grounded in a superior method. In fact, it happened that the group whose estimate was closest to the mark had based their guess on only one measurement, having also eliminated data from their sample based on a hunch about the result. In short, while important and powerful ideas surfaced through engaging with the Bigfoot MEA, these ideas were experienced as problematic and were not ready for complete "digestion" by the class. In particular, while it was clear that the function-based approach provided a wider view of the meaning of calculations, it was not so clear that this wider perspective was helpful in arriving at a "correct answer," and, in fact, the experience had called into question the very nature of correctness.

Given this class's experience with the Bigfoot MEA, the instructor selected a sequence of MDS activities that would offer further opportunities to explore Cartesian graph representations of linear and nonlinear relationships. We provide a brief account of each of these activities.

First, a Reflection Tool (RT) was used to support groups in reviewing and reflecting on their problem-solving processes. The instructor chose an RT format that used a qualitative graph of productivity over time (see Appendix C), with the intention of immediately extending and problematizing the Cartesian representation. The most common approach to "graphing" the groups' work over time was to divide the session into distinct periods when the group was following an idea or way of thinking. This supported the group in re-introducing the idea of a regime or sub-domain, which had surfaced in the relations between foot size and height in the Bigfoot problem.

Next, the instructor selected two Model Extension Activities (MXAs) that would provide opportunities to engage with the graphical representation of linear and non-linear functions, but in this case as *expressive* tools. In the first of these activities, Cell Growth
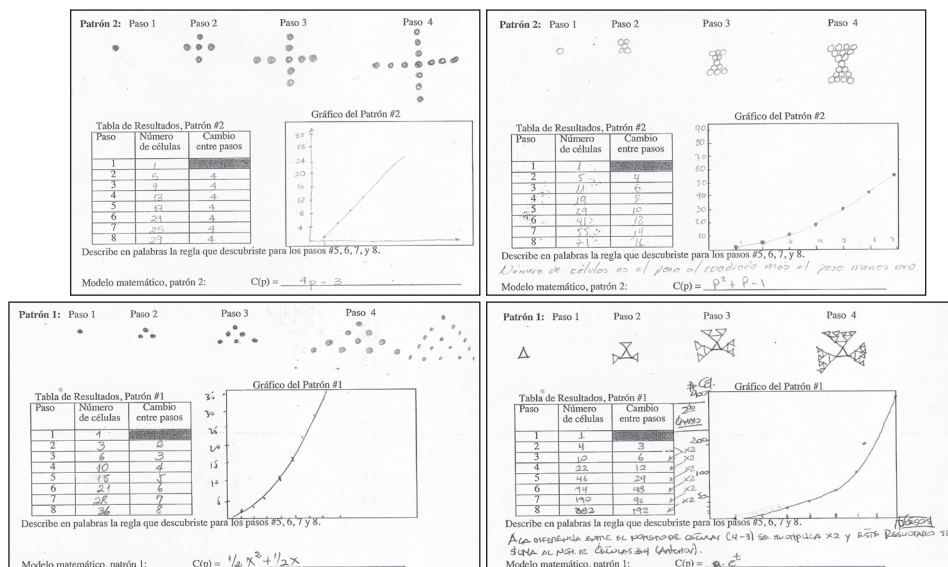
Figure 2. Work from the Cell Growth Pattern activity, showing linear (2a),
quadratic (2b,c), and exponential (2d) growth types.

Patterns (Stroup & Davis, 1999, and see Appendix D), learners created patterns of growth and described these patterns with tabular, graphical, and function representations.

Because these growth patterns were drawn in discrete steps, the resulting functions had integer-valued domains. Nevertheless, it appeared that the "scatterplot" of these values could be connected with an analytic function. However, having recently had an experience in which a function "fit" was problematic, some students reflected on the significance of the non-integer values of this graph.

In the second of the Model Extension Activities (MXAs), groups of students worked with calculator-based motion sensors to create linear and non-linear position-time graphs through bodily motion (c.f., Brady, 2013). For the linear case, the class recalled the story of the Tortoise and the Hare, in which the Tortoise wins by being "slow but steady." Class members "played tortoise" with several variations, creating graphs that the class as a whole could then model with linear functions. Having seen the "Tortoise's" motion enacted grounded the data in a physical experience, and yet the class found that there were still legitimate variations in modeling strategies. For instance, some focused on ensuring that the model agreed with the motion at the beginning and end of the race (see 3b); others chose to ignore the initial segment of the graph in which the "Tortoise" was stationary due to reaction time, modeling only the motion that occurred in the middle of the race (see 3c). The first strategy yielded a measure of average speed; the second expressed a notion of "typical" speed or "cruising speed."
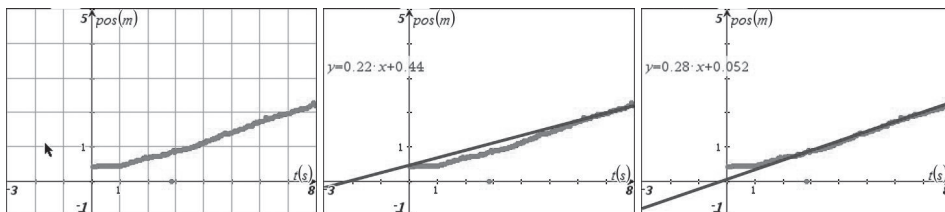
Figure 3. Graph of motion of the "Tortoise," and two linear functions modeling that motion.

The impact of these two modeling decisions was clear and even quantifiable, as the group had identified that the slope of the line constituted a measure of the Tortoise's speed in meters per second. And yet both models had justifications, depending on the purpose for which they would be used.

To experiment with motions that were non-uniform and functions that were not linear, in the next part of the MXA the groups created stories of motion and enacted these narratives, capturing them with the motion sensor. A selection is shown in figure 4. Each of these graphs was captured by motion detectors in groups. One "actor" enacted her or his motion story, while another group member operated the calculator and calculator-based motion sensor. This activity helped to reinforce the relationship between the graphical representation and the notion of change over time.
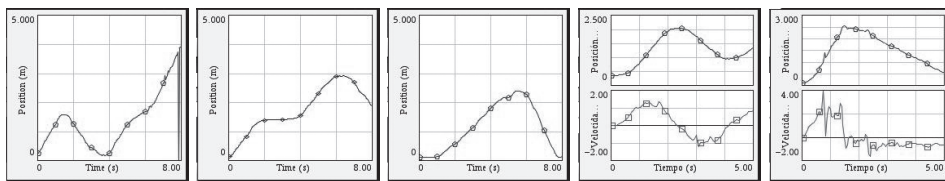


Figure 4. Examples of student-collected motion graphs: each is accompanied by a narrative.

As students iteratively developed their stories and enacted them with motion detectors, they became (1) more curious about the graphical representation itself, and (2) more expressive and inventive in manipulating it. For the curious, it was possible to turn on a velocity graph of the motion, as shown in figures 4d and 4e. Reflecting on the relations between the two graphs, and attempting to create a motion with a "clean" graph for both position and velocity, became a rich area of experimentation for these groups. And on the expressivity front, groups found ways of manipulating the data capture to achieve desired effects. For instance, in the story for graph 4c, a sudden change of direction and burst of speed was required. At first, the group could not achieve a sufficiently dramatic effect, because the actor could not physically accelerate sufficiently to produce the graph he desired; the group solved this problem by having the 'cameraman' *and* the actor move: at the point where the graph reaches its maximum, the actor and cameraman both moved toward each other.

In the final activity of this MDS, the group engaged with the Drills and Skills problem (Appendix E), which was positioned as a Model Adaptation Activity (MAA). In the problem, six different models are presented as fitting functions for a fictitious data set relating the time schools spent on drill and practice for a high-stakes exam to the average student score at the school on that exam. Each of the models has equal statistical validity, as measured by the sum of the "unexplained" variation over the data set. The students' challenge is to choose two of the graphs and provide a justification for each, explaining why this relation between practice and performance might hold.

In this activity, the teachers showed strong interpretive skills, both in giving a mechanistic account to support different fitting functions and in qualifying their interpretations/mechanisms by making reference to the high level of variation in the data. The relationship here between the data and the equation model was at least as problematic and challenging as in the Bigfoot data; the teachers' comfort in reasoning about this relationship was thus evidence of their having successfully internalized important concepts in this connection.
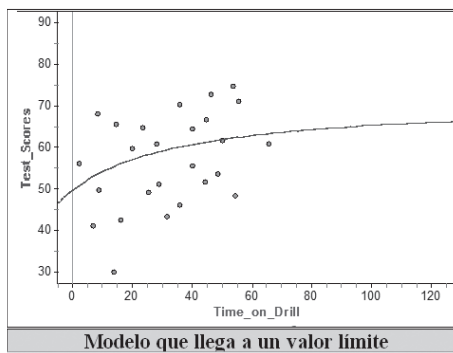


Modelo que llega a un valor límite



Modelo que aumenta y luego disminuye

The student begins from prior knowledge or from initial learning. Beginning from there, his achievement grows as a function of time, until it reaches a maximum point, after which the achievement level stabilizes and becomes constant, that is to say, if he practices more, he no longer sees an improvement in results; that is, now he needs to work on other types of activities to further improve outcomes.
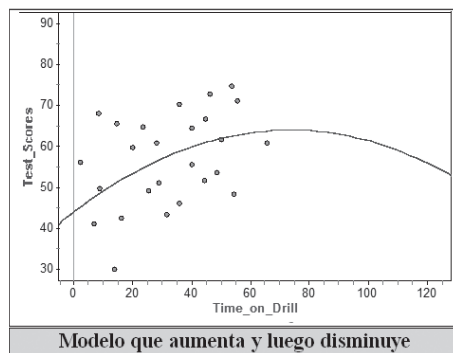
The student begins from prior knowledge or from initial learning. Beginning from there, his achievement grows as a function of time considerably, until it reaches a maximum point, after which his achievement begins to diminish, due to fatigue or being overwhelmed by the required activity of practicing.

Figure 5. Two groups' interpretations of explanatory models for the Drills & Skills data (translations by the instructor).

Thus, over the course of this MDS sequence, a primary problem-solving experience with an MEA (Bigfoot) gave rise to debate about the use of functions in making inferences

based on data. Key ideas about a function's domain and the way the function expresses covariation became problematic for the group. This debate raised fundamental questions about the strengths and limits of linear models and how they connect to and support arithmetic procedures. This setting opened the way for the activities that more openly explored of the meaning and "workings" of Cartesian graphs, which was pursued with a Reflection Tool and in two Model Exploration Activities. Finally, teachers were then able to apply an enhanced sensibility for function graphs to a Model Adaptation Activity that foregrounded the range of possible interpretations, mechanisms, and models which could underlie a curve-fitting argument.

## Illuminating Design and Implementation Principles for MDS activities

During the course module, the first author was conscious of the goals and decisions that guided his selection of activities for the MDS. In discussing this work with the author team, the group identified common themes between these decisions and other MDS work, enabling us to articulate them as provisional design principles. The central instructional situation here was that a classroom group had engaged with an MEA and had developed solutions to the client's problem that they deemed satisfactory. Later, however, in sharing out their solutions, they recognized limitations in their thinking and in the models they have produced. This situation is not a universal one, but it has occurred often in our experience of working with MEAs. Principles for MDS sequences are intended to resonate with instructional goals appropriate to situations that are typical of the state of idea development found in classes after MEAs. Below, we provisionally lay out six principles that emerged in our discussion of this episode, as we discussed and analyzed it in the light of our broader experience in designing and implementing MDS sequences For each principle, we discuss the way that it appeared in the first author's instructional decision-making.

　　**1. The Experiencing Multiple Perspectives principle.** Activities in an MDS should provide learners with opportunities to see the core mathematical concepts that have surfaced in their MEA work from multiple perspectives. This begins with the MEA itself, as groups share their solutions informally, in presentations, or in "poster sessions." Variation across groups prompts learners to engage in additional levels of reflection, as they experience different ways of thinking about closely related mathematical constructs. Then, in MDS activities, students should be given the opportunity to see how related but different assumptions and problem situations call for different but related models and solutions to the ones they have developed in MEAs. This Experiencing Multiple Perspectives principle is somewhat analogous to the Generalizability principle for MEAs (see above); and it is also related to the Multiple Embodiments principle that was articulated by Zoltan Dienes (*e.g.*, 1971).

　　In the episode described above, multiple perspectives were powerfully used in several ways. First, the Cartesian representation itself entered the discussion as a means of providing an alternative perspective on the Bigfoot problem, and Cartesian graphs were subsequently encountered in a variety of settings. Moreover, the different activities involved

different relations between functions and the data they were intended to represent. In the Cell Growth Patterns activity, many students were able to fit analytic functions exactly to their data, so that the function seemed to express the "internal logic" of the growth pattern, even extending that logic to a continuous rather than discrete view of growth. In contrast, in the Tortoise motion-sensor activity, differences in modeling choices came to the forefront, emphasizing the interpretive role of the match between function and data in this setting. And finally, in the Drills and Skills problem, the functions that were "fit" to the data explicitly involved *imposing* interpretations on data that were designed to admit several statistically valid interpretations. This line of inquiry built upon the class's realization during the Bigfoot MEA that their proportional approaches implied a model of human growth that was based on a linear function passing through the origin. It gave them a context for reflecting on such models in the broader context of functions that provide alternative and competing *explanatory narratives* for phenomena represented in datasets.

**2. The Reflectively Processing Shared Experiences principle.** M&MP perspectives on learning and idea development tend to value the personal and even idiosyncratic connections that learners may make as they solve problems. It is thus extremely important to give groups opportunities to reflect on the way that they have solved problems. In addition, at the level of instructional design it is desirable to shape learning sequences so that *they* are responsive to the challenges, questions, and concerns raised by the learners themselves and offer them opportunities to process the ideas that have emerged for them. To follow this principle, MDS activities should thus sustain an extended conversation with the questions that were encountered by the groups as important to their work. A key issue here is "Who 'owns' the mathematics that we are discussing?" Ideally, the answer is "the students" — that is, that the investigations in the MDS are grounded in questions that they themselves have raised implicitly or explicitly. In the episode described here, this principle appeared in the instructor's decision to depart from his prepared instructional sequence in order to give greater emphasis to graphical representations and the interpretive significance of matching functions to data. Moreover, it also surfaced in the instructor's use of generative activities that allowed learners to construct mathematical objects that expressed their own particular interests and ways of thinking (Stroup, Ares, Hurford, & Lesh, 2007; Stroup, Ares, & Hurford, 2005). .

**3. The Exploring the Range of Motion of Ideas principle.** According to this principle, the situations we model in realistic problem solving are often dynamic systems, producing exhibiting a range of different phenomena under variations in key parameters. In modeling such systems, it is important to capture these variations, so as to be able make decisions where parameters may change or may be uncertain. Modeling in real-world situations often involves optimization and tradeoffs, so it is rarely sufficient to comprehend a system through a "snapshot" of its behavior. Fortunately, as both learners and instructional designers we have at our disposal an increasing range of computational environments for realizing our models in dynamic, executable form. MDS activities that offer such representational environments to learners can support extensions of the think-

ing that groups were able to achieve in the time-compressed experience of MEAs. In our case study, this principle emerged first in the discussion of the Bigfoot MEA. Realizing the measurements taken by different groups in scatterplots and interpreting the "rule of three" in terms of a linear function through the origin, groups could visualize the effects of changes in the data points, the effects of measurement errors of different sizes (a concern of one group in particular), or the challenges that the cub scouts might encounter as children attempting to implement the groups' (adult-based) procedures.

**4. The Building Representational Fluency principle.** This principle recognizes the central role played by representations in modeling, not only in *communicating* ideas but also in supporting thinking itself. Thus, many MDS activities may focus on learning the craft of disciplinary uses of representations as expressive, versatile, and powerful tools. This principle acted as a primary motivator for activity selection in the instructional sequence of our case, as all of the activities emphasized using Cartesian space as an expressive medium for meaning-making.

**5. The Supporting Authentic Formalization principle.** While Principle #2 emphasizes that MDS activities should remain closely tied to authentic questions and ideas generated by groups in their MEA solutions, this principle holds that it is also valuable for learners to connect their solutions and ways of thinking with the practices of the larger mathematical community. Activities that introduce conventional techniques and algorithms as well as standard terminology may be responding to this principle. In the episode discussed here, this principle was under-represented: indeed, the instructional decision to emphasize representational fluency was made at the expense of activities that would have explored the notion and mathematical basis of statistical measures of fit for linear models.

**6. The Recognizing Mathematization as Active Interpretation principle.** This principle emphasizes that modeling in authentic settings goes beyond applications to involve fundamental acts of interpreting the world. Activities attending to this principle may foreground the consequences of applying given models as lenses on phenomena, including opportunities to "go beyond thinking *with* a given model to also think *about* it" (Lesh et. at 2003, p39).

This principle surfaced at various points in the implementation episode. In the discussion of the Bigfoot MEA itself, this principle was behind the move made to rethink the mathematical basis for the rule of three and the implications of applying this ratio-based approach to infer the height of Bigfoot. In the Cell Growth Patterns activity, it was active in discussions where students classified patterns in groups. Some patterns that appeared dissimilar on the surface were later seen as being in the same "family" (*e.g.*, linear, quadratic, or exponential types of growth). At the same time, some growth patterns that seemed similar on the surface were later identified as belonging to different growth families. And finally, in the Drills and Skills Model Adaptation Activity, this principle was involved in the fact that one can "see the data as" expressing and being modeled by a variety of very different function-types, each of which carries a predictive story about the relationship between additional practice time and student achievement.

## Conclusion

In this article, we have argued that the practice of mathematical problem solving in the world outside of school has changed dramatically in recent years. This situation constitutes an urgent call for change in the practices of school mathematics — both in the mathematical *tools* that students use and the kinds of mathematical modeling and problem solving *practices* that students engage in. At the same time, we argued that the school setting has important features absent from real-world problem-solving settings, which should be exploited in the design of instructional sequences around MEAs. We introduced the idea of a Model Development Sequence as it has emerged in recent design research within the M&MP tradition. We emphasized the importance of maintaining the adaptability and flexibility of MDS materials, recognizing the teacher's need to construct activity sequences that are responsive to the needs of their own particular classroom groups. Nevertheless, we showed that this selection and adaptation process is a highly purposeful activity, which can benefit from guiding principles. Then, we presented an episode from a recent implementation experience, where the instructor assembled an activity sequence around the Bigfoot problem, responding to the group's inquiry and questions in working with that MEA. We described the MDS that resulted, and we articulated six provisional principles for assembling MDS sequences, which emerged from discussing the episode in the context of the author group's broader experience with MDSs.

We argue that MDS sequences built around MEAs provide an important combination of (1) the authenticity of problem solving as it is experienced in the world outside of school and (2) the opportunities for reflection, unpacking and formalization that can be achieved through carefully designed and adapted extension activities. On one hand, MEAs offer problem solving situations that put learners off balance and require them to adapt their formal mathematical knowledge creatively. On the other hand, MDS activities help learners to appropriate and "domesticate" the raw ideas that emerge in their MEA work, as well as to relate these ideas to standard disciplinary representations, procedures, and practices. By combining these two aspects, we can create learning environments that act as authentic simulations of real-world problem solving while also providing ample opportunities to reflect, unpack, and formalize important concepts in the curriculum.

### Note

[1]  One group even decided to sample *only* the instructor, reasoning that there might be culturally-specific foot-size to height ratios, and since the problem came from North America, they should restrict their sample to only North Americans.

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## Appendix A: The Summer Jobs problem

### *The Summer Jobs Problem*

Last summer Carla started a concession business at Wild Days Amusement Park. Her vendors carried candy, hot dogs, and drinks around the park, selling wherever they found customers.

The business was a great success. Next summer, Carla is expecting that all of her vendors will want to work for her again. But, the park managers told her that she won't be allowed to hire as many vendors next summer. So, she needs your help deciding which workers to rehire. If all of last year's vendors apply for a job, she'll only be able to hire about a third of them to work full time, and about a third of tehm to work half time. She won't be able to hire the remaining third of them.

The table below shows a sample of nine people who worked for her last summer. To try to figure out a procedure for deciding who to hire next summer. Carla reviewed her records for the nine vendors who are shown. For each of these vendors, she totaled the number of hours they worked and the amount of money collected – when business in the park was busy (high attendance), steady (average attendance), and slow (low attendance). (See the table that follows.) She wants to rehire the vendors who will make the most money for her. But, she doesn't know how to compare them because they worked different numbers of hours; and, she isn't sure what to do about the fact that it's easier to sell more when the attendance is high.

Write a letter to Carla describing how she can evaluate all of the vendors who worked for her last summer, and how to decide who to hire full-time and part-time. Show how your procedure works for the nine people workers who are shown in the table. Give details so Carla can check your work, and give a clear explanation so she can decide whether your method is a good one for her to use.

| HOURS WORKED LAST SUMMER | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | JUNE | | | JULY | | | AUGUST | | |
| | Busy | Steady | Slow | Busy | Steady | Slow | Busy | Steady | Slow |
| MARIA | 12.5 | 15 | 9 | 10 | 14 | 17.5 | 12.5 | 33.5 | 35 |
| KIM | 5.5 | 22 | 15.5 | 53.5 | 40 | 15.5 | 50 | 14 | 23.5 |
| TERRY | 12 | 17 | 14.5 | 20 | 25 | 21.5 | 19.5 | 20.5 | 24.5 |
| JOSE | 19.5 | 30.5 | 34 | 20 | 31 | 14 | 22 | 19.5 | 36 |
| CHAD | 19.5 | 26 | 0 | 36 | 15.5 | 27 | 30 | 24 | 4.5 |
| CHERI | 13 | 4.5 | 12 | 33.5 | 37.5 | 6.5 | 16 | 24 | 16.5 |
| ROBIN | 26.5 | 43.5 | 27 | 67 | 26 | 3 | 41.5 | 58 | 5.5 |
| TONY | 7.5 | 16 | 25 | 16 | 45.5 | 51 | 7.5 | 42 | 84 |
| WILLY | 0 | 3 | 4.5 | 38 | 17.5 | 39 | 37 | 22 | 12 |
| MONEY COLLECTED LAST SUMMER (IN DOLLARS) | | | | | | | | | |
| | JUNE | | | JULY | | | AUGUST | | |
| | Busy | Steady | Slow | Busy | Steady | Slow | Busy | Steady | Slow |
| MARIA | 690 | 780 | 452 | 699 | 758 | 835 | 788 | 1732 | 1462 |
| KIM | 474 | 874 | 406 | 4612 | 2032 | 477 | 4500 | 834 | 712 |
| TERRY | 1047 | 667 | 284 | 1389 | 804 | 450 | 1062 | 806 | 491 |
| JOSE | 1263 | 1188 | 765 | 1584 | 1668 | 449 | 1822 | 1276 | 1358 |
| CHAD | 1264 | 1172 | 0 | 2477 | 681 | 548 | 1923 | 1130 | 89 |
| CHERI | 1115 | 278 | 574 | 2972 | 2399 | 231 | 1322 | 1594 | 577 |
| ROBIN | 2253 | 1702 | 610 | 4470 | 993 | 75 | 2754 | 2327 | 87 |
| TONY | 550 | 903 | 928 | 1296 | 2360 | 2610 | 615 | 2184 | 2518 |
| WILLY | 0 | 125 | 64 | 3073 | 767 | 768 | 3005 | 1253 | 253 |

Figures are given for times when park attendance was high (busy), medium (steady), and low (slow).

## Appendix B: The Bigfoot problem

**The Bigfoot Problem**

A cub scout troop from Saint Paul has been doing a project at a local park. Last weekend, they noticed as they were leaving that the old brick drinking fountain had started to leak. They mentioned the problem to a Park Ranger, who shut it off.

This week, they are back. Arriving early in the morning, they noticed that the drinking fountain had been fixed. But this was a surprise to the Ranger: Apparently someone had fixed it after the park closing time the night before!

The scouts and the Ranger agree that it would be great to be able to thank the people who fixed the water fountain. But all they could find were lots of muddy footprints.

One set of footprints was distinctive – extremely large. A full-sized model of these footprints are taped on the floor of the classroom in the same position that they were found. The scouts and the Ranger feel that this is the person they would like to know more about. To find this person and his or her friends/associates, it would help if they could figure out how big he or she really is.

**Your task** is to help the scouts by making a "how to toolkit" — a step by step procedure that they can use to figure out how big people are by looking at the footprints they leave behind. Your toolkit should work for footprints like the ones that the group has found, but it should also work for other footprints.

## Appendix C: A Group Reflection Tool

**Group Reflection Tool: Your Group's Problem-Solving.**

1. *Graph the Session.* Take 5 minutes to discuss your group's problem-solving work. Make a "Graph of Progress" that describes how the group moved toward its final solution. In the example below, notice that the graph doesn't always show forward progress: sometimes the group felt they were making no progress, and sometimes they felt they were moving away from a solution.

**A Graph of Progress for a Solution to a Modeling Activity**
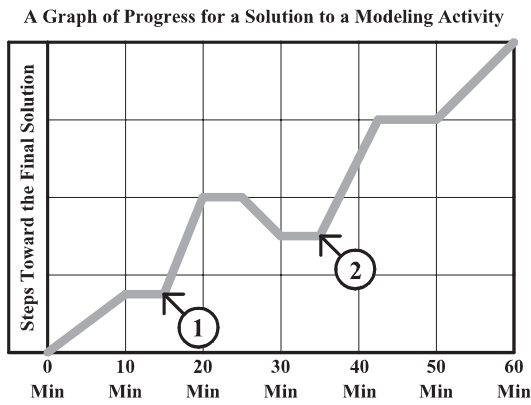


In the space below, sketch your group's problem solving graph. Discuss this as a group until you all agree on a graph that represents your work.

*Your Group's Problem Solving Graph and Two Critical Points*

2. *Identify Two Critical Points.* In thinking back on the group's problem-solving session, identify two critical moments in the session, when the group changed its rhythm. Mark them on your group's problem solving graph, above.

# Appendix D: The Cell Growth Patterns activity

**Creating Cell Growth Patterns**

In this activity (on the back of this page), you will create two different growth patterns of your own. For each pattern, you'll draw out the first 4 steps, and you'll fill in a table showing how the patterns continue numerically. A good "check" on whether your pattern is clear will be whether the other members of your group can extend the pattern based on your drawings of Steps 1-4.

Here is an example we can go through together.
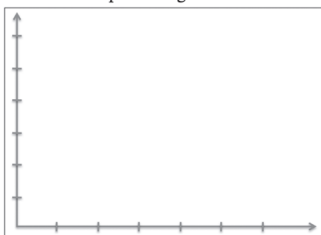
*Example:*

Step 1        Step 2        Step 3              *Now try drawing step 4:*

Table of values, Example Pattern

| Step | Number of 'cells' | Change between Steps |
|------|-------------------|----------------------|
| 1    |                   |                      |
| 2    |                   |                      |
| 3    |                   |                      |
| 4    |                   |                      |
| 5    |                   |                      |
| 6    |                   |                      |
| 7    |                   |                      |
| 8    |                   |                      |

Graph of the growth:

Try to describe in words how the pattern grows for steps #5, 6, 7 and 8:

Now try to create a mathematical rule that will give you the value of "Number of 'cells'" based on the "step"

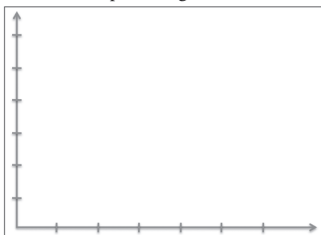Cells(s) = _____

**Your Pattern #1:**

Step 1              Step 2              Step 3              Step 4

Table of values, Pattern #1

| Step | Number of 'cells' | Change between Steps |
|------|-------------------|----------------------|
| 1    |                   |                      |
| 2    |                   |                      |
| 3    |                   |                      |
| 4    |                   |                      |
| 5    |                   |                      |
| 6    |                   |                      |
| 7    |                   |                      |
| 8    |                   |                      |

Graph of the growth in Pattern #1

Try to describe in words how the pattern grows for steps #5, 6, 7 and 8:

Mathematical Model for Pattern 1:        Cells(s) = _____

## Appendix E: The Drills & Skills problem

In two school districts in Illinois, the superintendents and school board members are considering a new policy, which they hope will help raise their students' scores on the state's end-of-year standardized test. The new policy will require all teachers to increase the amount of time that they spend each week on drills, which focus on the kind of basic skills that the tests emphasize.

Table 1 shows information about the past performance of students in the 26 middle schools where this new policy will be enforced. For each school, the table shows the average amount of time that each school spent on drill each week. Then, the table also shows each school's average score on the end-of-year test. Questionnaires and observations were used to estimate the current amount of time that teachers spend on drill each week.

Before the new policy is adopted, the school boards have asked anybody who is interested to submit a brief letter (no more than 2-pages) explaining their predictions about what will happen if all schools are required to spend 90 minutes per week on drills.
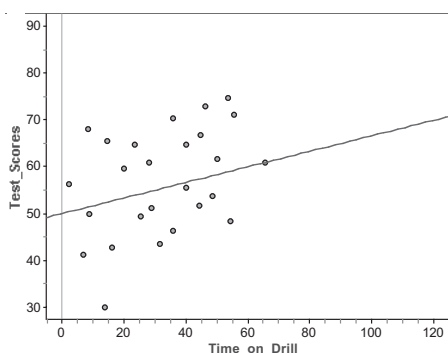
Unfortunately, even though the letters that the school board received gave lots of opinions, they didn't give very good explanations. Several of the predictions that seemed to be most worthwhile to consider included graphs like those shown in Table 2. But, only one of these graphs came with a "story" that explained why the trend and prediction was more sensible than others that were suggested. And, the school district's data analysis person discovered that it didn't help to ask: "Which graph fit the data best?" – because all six graphs fit the data equally well. That is, the sum of distances from the prediction line to the points on the graph were exactly the same for each of the six prediction lines! So, the prediction that the school board will believe probably will be the one that is accompanied by a convincing "story" that explains the trend.

Your Task: Please write "stories", similar to the one given below, that explain the other five trends and predictions.

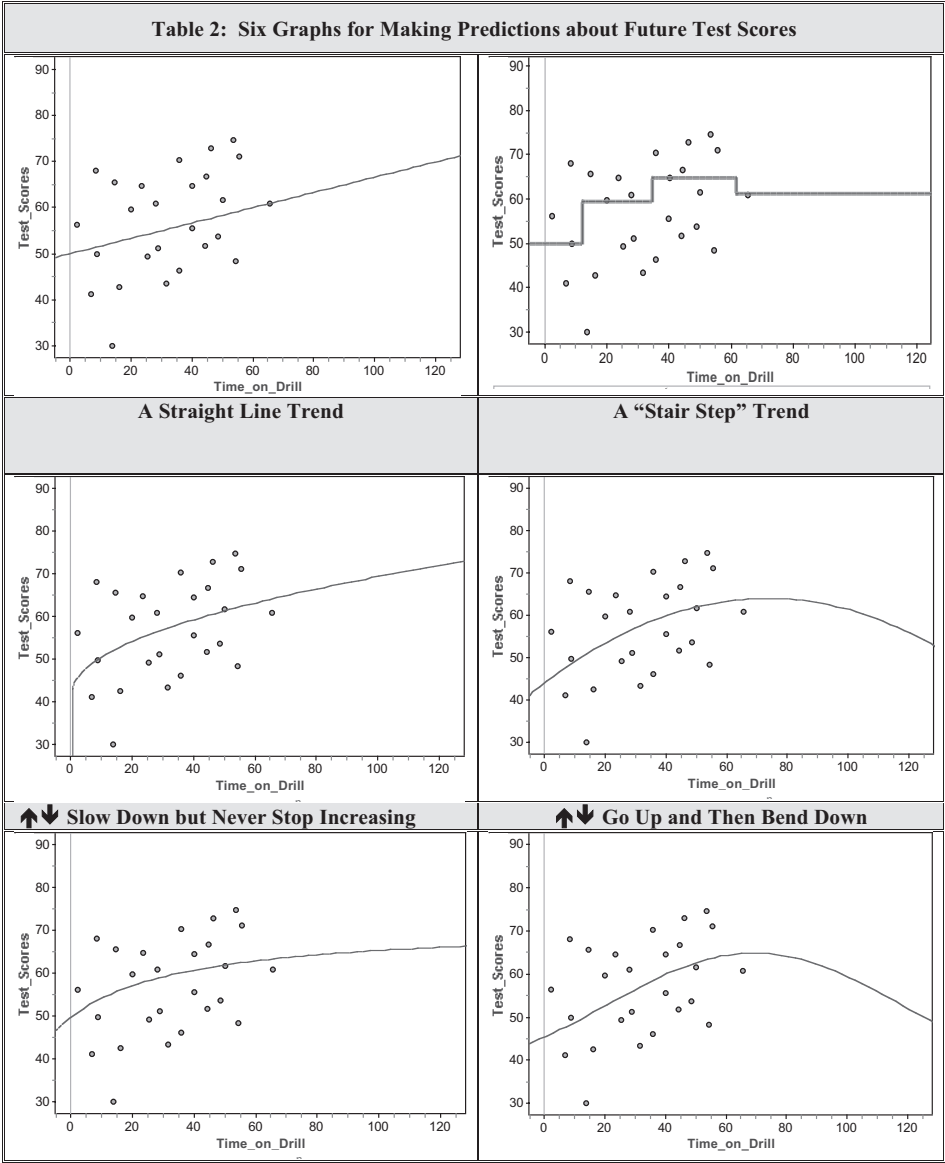| Table 1: Time Spent on Drill as a Predictor of Performance on a Standardized Test | | |
|---|---|---|
| **School** | **Average Time Spent on Drill Each Week** | **Test Scores** |
| A | 17.49 | 49.55 |
| B | 28.02 | 48.76 |
| C | 23.62 | 64.69 |
| D | 36.6 | 57.74 |
| E | 39.92 | 55.55 |
| F | 44.54 | 69.96 |
| G | 19.53 | 55.63 |
| H | 40.22 | 68.92 |
| I | 48.77 | 53.69 |
| J | 14.18 | 58.79 |
| K | 20.54 | 68.97 |
| L | 25.06 | 42.02 |
| M | 26.95 | 66.37 |
| N | 40.09 | 59.86 |
| O | 25.88 | 58.51 |
| P | 28.95 | 54.99 |
| Q | 23.79 | 48.68 |
| R | 34.17 | 53.84 |
| S | 49.29 | 59.75 |
| T | 35.92 | 70.34 |
| U | 46.09 | 72.9 |
| V | 52.83 | 55.93 |
| W | 54.46 | 61.24 |
| X | 53.45 | 74.69 |
| Y | 55.53 | 71.11 |
| Z | 66.4 | 60.76 |

**Dear School Board Members,**

**For every extra minute that students spend on drill, their test score should be expected to increase at a steady rate. So, as more time is spent of drill, the trend should look like a straight line. For example, if students spend 90 minutes on drill, then my graph shows that the average test scores should be about 65.**

**Sincerely,**
**Sam Straight**

**Table 2:  Six Graphs for Making Predictions about Future Test Scores**



**A Straight Line Trend**

**A "Stair Step" Trend**

⬆⬇ **Slow Down but Never Stop Increasing**

⬆⬇ **Go Up and Then Bend Down**

**Resumo.** Neste artigo, abordamos o desafio de apoiar os alunos na resolução de problemas da vida real como uma parte estimulante da sua experiência de aprendizagem da Matemática na escola. Por um lado, descrevemos os aspetos controversos ligados às abordagens escolares tradicionais na preparação dos alunos para o sucesso num mundo cada vez mais caraterizado pela complexidade e pela rápida mudança das áreas disciplinares. Atualmente os profissionais com formação nos domínios STEM (science, technology, engineering, and mathematics) precisam de adaptar, de forma criativa, o conhecimento que obtiveram na escola para o usarem de modo eficaz; assim, é desejável propor aos alunos atividades que ofereçam experiências mais realistas em cenários de resolução de problemas para além da escola. Por outro lado, defendemos que o contexto escolar proporciona oportunidades únicas para a reflexão em torno de experiências de resolução de problemas, permitindo expor, ampliar e formalizar as ideias que emergem das resoluções dos alunos em problemas realistas. Para harmonizar estas duas necessidades, apresentamos as Atividades Geradoras de Modelos (Model-Eliciting Activities — MEAs) como contextos autênticos de resolução de problemas e descrevemos as Sequências de Desenvolvimento de Modelos (Model-Development Sequences – MDSs) como uma estrutura para atividades de aprofundamento que ajudem os alunos a processar os modelos conceptuais que produziram nas MEAs e levem os vários grupos na sala de aula a formular ideias partilhadas, consistentes com as normas e convenções da disciplina. Tendo por base esta estrutura de trabalho, apresentamos um episódio de ensino, extraído de uma pesquisa de *design-research* em curso, no qual o primeiro autor elaborou e adaptou uma MDS em resposta às necessidades dos alunos que emergiram no momento. Concluímos com a descrição das decisões didáticas que guiaram o nosso trabalho e formulamos um conjunto de seis princípios de *design* de MDSs que sobressaíram da nossa análise reflexiva deste episódio de implementação.

*Palavras-Chave:* Modelação Matemática, Resolução de Problemas, Problemas do Mundo Real, Princípios de *Design*.

**Abstract.** In this article, we address the challenge of supporting students in real-world problem solving as a vibrant part of their in-school experience of mathematics. On one hand, we describe the issues associated with using traditional schooling approaches to prepare students for success in a world that is increasingly characterized by complexity and rapid disciplinary change. Today's STEM professionals need to adapt the knowledge they have learned in school in creative ways to use them effectively; thus it is desirable to provide learners with activities that offer a more realistic simulation of problem-solving settings beyond school. On the other hand, we argue that the schooling context does offer unique opportunities for reflecting on problem-solving experiences and for unpacking, extending, and formalizing ideas that emerge in students' solutions to realistic problems. To balance these two needs, we describe Model-Eliciting Activities (MEAs) as authentic problem-solving settings, and we describe Model-Development Sequences (MDSs) as a framework for extension activities that help students to process the conceptual work they have done on MEAs and that help classroom groups to develop shared understandings that are consistent with disciplinary norms and conventions. With this frame in place, we then present a teaching episode extracted from ongoing design research, in which the first author elaborated and adapted an MDS in response to student needs that surfaced in the moment. We describe the instructional decisions that guided this work, and we articulate a set of six design principles for MDS sequences, which emerged through the author group's reflective analysis of this implementation episode.

*Keywords:* Mathematical Modeling, Problem Solving, Real-World Problems, Design Principles

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