Analyzing teacher learning in lesson study: mathematical knowledge for teaching and levels of teacher activity

Analisando a aprendizagem do professor num estudo de aula: conhecimento matemático para ensinar e níveis de atividade do professor

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Abstract. In this paper, we analyze and detail the knowledge incorporated by mathematics teachers in their participation in lesson study. Utilizing an extended theoretical framework of mathematics teacher knowledge, combining the cognitive frameworks of Mathematical Knowledge for Teaching and Levels of Teacher Activity, we detail the knowledge articulated in teachers' collaborative lesson study conversations. This theoretical framework is presented as a theoretical tool to detail the incorporation of mathematics teachers' knowledge in planning, conducting, and reflecting on research lessons in lesson study. Using data generated in two case studies, one from the Republic of Ireland (post-primary) and another from Switzerland (primary), we examine the knowledge articulated by mathematics teachers in their planning and reflection of a research lesson. We detail this knowledge using qualitative excerpts and provide quantitative analysis of the knowledge categories incorporated by teachers in each phase of a lesson study cycle. Our analysis provides evidence that, in their participation in lesson study, teachers draw on and incorporate all elements of their mathematical knowledge for teaching at all levels of teacher activity when planning and reflecting on a research lesson.

Keywords: teacher collaboration; lesson study; professional development; mathematical knowledge for teaching; mathematics education.

Resumo. No presente artigo analisamos e detalhamos o conhecimento incorporado por professores de Matemática na sua participação num estudo de aula. Utilizando um quadro teórico alargado de conhecimento do professor de Matemática, combinando as estruturas cognitivas do Conhecimento Matemático para Ensinar e Níveis de Atividade do Professor, detalhamos o conhecimento articulado em conversas ocorridas entre os professores no estudo de aula colaborativo. Este quadro teórico é apresentado como uma ferramenta teórica para detalhar a incorporação do conhecimento dos professores de Matemática no planeamento, condução e reflexão sobre as aulas de investigação de estudos de aula. Usando dados gerados em dois estudos de caso, um da República da Irlanda (nível pós-primário) e outro da Suíça (nível primário), examinamos o conhecimento articulado por professores de Matemática no seu planeamento e reflexão sobre uma aula de investigação. Detalhamos este conhecimento usando excertos qualitativos e fazemos uma análise quantitativa das categorias de conhecimento incorporadas pelos professores em cada fase do ciclo de um estudo de aula. A nossa análise fornece evidências de que, na sua participação no estudo de aula, os professores baseiam-se e incorporam todos os elementos de seu conhecimento matemático para ensinar em todos os níveis de atividade de professor, ao planear e refletir sobre uma aula de investigação.

Palavras-chave: colaboração docente; estudo de aula; desenvolvimento profissional; conhecimento matemático para ensinar; educação matemática.

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Introduction

Teaching mathematics requires a complex set of knowledge and skills. Over the past two decades, much international research has focused on detailing the expertise required to teach mathematics, in both large-scale studies (e.g. Baumert et al., 2010; Hill, Ball, & Schilling, 2008) and smaller-scale, classroom based research (Clivaz, 2017; Margolinas et al., 2005). Carefully defining and theorizing the knowledge required to teach mathematics is key to understanding how such knowledge can be developed in teacher education, at both pre-service and in-service stages. However, there are diverse perspectives of the knowledge required to teach mathematics across different mathematics education research traditions and variations in these perspectives have impacted the methodological designs and findings of research studies across cultures (Depaepe, Verschaffel, & Kelchtermans, 2013).

Of the vast array of research related to developing mathematics teachers' knowledge, lesson study is one model which has grown in popularity across the globe (Huang & Shimizu, 2016). A number of studies have demonstrated the potential of lesson study to develop teacher knowledge (e.g. Lewis & Perry, 2017; Ni Shuilleabhain, 2016), impact classroom practices (e.g. Olsen, White, & Sparrow, 2011; Takahashi, 2014) and build teacher community (Baricaua Gutierez, 2016; Cajkler, Wood, Norton, Peddar, & Xu, 2015). However, with such increased attention of international educational research on lesson study, there have also been calls to deepen the understanding of the development of mathematics teacher knowledge within this model (Lewis, Perry, & Hurd, 2009) and to provide a solid theoretical foundation for its use in teacher education (Clivaz, 2015b; Miyakawa & Winslow, 2009).

Teacher knowledge is often separated into categories of: content knowledge and pedagogical content knowledge (Grossman, 1995; Shulman, 1986). Teachers' knowledge of subject matter is a necessary pre-requisite for their profession and has been found to positively influence students' learning of mathematics (Baumert et al., 2010; Hill, 2010; Ma, 1999). However, in considering pedagogical content knowledge (PCK), debate has arisen in the categorization and construct of this form of knowledge and its impact on student learning. The Anglo-American construct of PCK (Depaepe et al., 2013) has been defined as that knowledge category "most likely to distinguish the understanding of the content specialist from that of the pedagogue" (Shulman, 1986, p. 8). This perspective of PCK is closely related to *didactique* in the French-speaking traditions or *fachdidaktik* in German-speaking traditions. However, in the theorization of PCK across various cultures, there are important differences in the interpretations of this knowledge as a factual knowledge, existing separately from other types of knowledge required by the teacher (cognitive), or knowledge inherently linked to and situated in the act of teaching in a particular context (situated) (Hodgen, 2011; Petrou & Goulding, 2011). A cognitive perspective on PCK has greatly contributed to mathematics teacher education in, for example, providing empirically based evidence of a positive correlation between this particular form of teacher knowledge and student learning (e.g. Baumert et al., 2010; Hill, 2010). However, some have criticized this perspective of such knowledge as being "static" and removed from the in-the-moment practices of the classroom (e.g. Petrou & Goulding, 2011). A more situated perspective on PCK considers this knowledge as dynamic and relevant to specific classroom context (Blanco, 2004; Seymour & Lehrer, 2006). Such a perspective explicitly values the classroom practices and teacher beliefs, which positively contribute to student learning (Hodgen, 2011). However, by only observing classroom practice, it is difficult to unravel this knowledge from other forms of knowledge enmeshed in the decisions and actions that teachers make. While these varying perspectives of PCK have contributed to our understanding of the knowledge required to teach mathematics, there remains a divide in how we consider and investigate the knowledge required to teach the subject (Depaepe et al., 2013) and there have been calls to further detail and comparatively analyze these frameworks (Rowland & Ruthven, 2011; Speer, King, & Howell, 2015).

In this paper, we hope to contribute to the literature by analyzing and detailing the knowledge incorporated by mathematics teachers in their participation in lesson study. We propose a theoretical framework, incorporating both the cognitive and situated perspectives on teacher knowledge, and utilize it to provide a fine-grained analysis of the knowledge included by mathematics teachers in their participation in lesson study. We seek to deliberately build on previous existing frameworks of mathematics teacher knowledge and propose this framework as a combination, in the sense of Prediger, Bikner-Ahsbahs, and Arzarello (2008), of the exising frameworks of Mathematical Knowledge for Teaching (Ball et al., 2008) and the Levels of Teacher Activity (Margolinas et al., 2005) to encompass the breadth and depth of mathematics teacher knowledge incorporated in lesson study. We analyze data generated in two case study sites and provide qualitative and quantitative analyses of the knowledge levels and types incorporated by teachers over one cycle of lesson study.

Lesson study

Lesson study is a collaborative model of professional development which supports teacher learning (Huang & Shimizu, 2016). Originating in Japan, this model has grown in popularity over the past two decades and is now being incorporated across the globe, from Korea to Uganda and from the USA to Portugal (Ponte, Quaresma, Baptista, & Mata-Pereira, 2013; Fujii, 2014; Stigler & Hiebert, 2016). Lesson study and research on lesson study have become particularly prevalent in the field of mathematics education and much research has detailed evidence of mathematics teacher learning through lesson study across the world (e.g. Doig, Groves, & Fujii, 2011; Dudley, 2013; Lewis et al., 2009; Lim, Kor, & Chia, 2016; Ni Shuilleabhain & Seery, 2017; Ono & Ferreira, 2010). Lesson study provides teachers with opportunity to contextualize representations of their classroom activities, while also making their implicit knowledge and practices explicit through collaborative conversations (Fujii, 2016). Participating in this model of professional development provides teachers with the opportunity to articulate, share and develop their knowledge, where student learning is at the core of their activities.

Teacher learning is best supported when teachers are active in their involvement, where there is a focus on content, and when teachers have opportunity to reflect on their own and students' learning (Murata, 2011). In addition, teacher professional development has greater potential to impact teacher learning when it involves the collective participation of a number of teachers and occurs over a period of time (i.e., not just one day) (Desimone, 2009). In lesson study, each cycle usually occurs over a number of weeks and consists of several steps where teachers begin by studying the curriculum and deciding on a research theme. Teachers then collectively plan a research lesson according to that theme, conduct and observe a live research lesson, and reflect on student learning within the lesson (see Figure 1) (Lewis, Perry, & Murata, 2006).

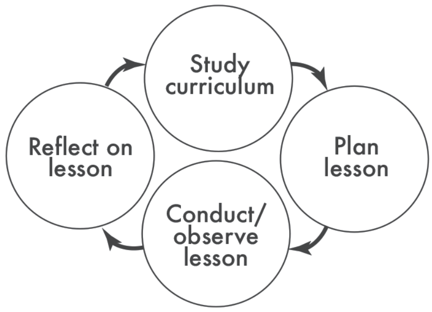

Figure 1. Lesson Study Cycle based on Lewis et al. (2006)

While the cycle, in itself, may seem simple and straightforward, the lesson study model provides a powerful means through which teachers can be supported in researching and developing their own practice, by focusing and reflecting on student thinking and on content-specific pedagogical activities (Fernandez, Cannon, & Chokshi, 2003; Meyer & Wilkerson, 2011; Takahashi & McDougal, 2016). Participants are provided with a window to critically review teaching and learning, at both a global and local level (Meyer & Wilkerson, 2011), and, through their collaboration with colleagues, have opportunity to build new knowledge, share learnings, and introduce new practices related to teaching and learning (Dudley, 2013; Ni Shuilleabhain, 2016; Takahashi, 2014).

While it is important to note that teaching and learning are cultural activities and that it may be difficult to transport lesson study across different cultures and education systems (Robutti et al., 2016; Stigler & Hiebert, 2016), research has evidenced teacher learning in lesson study in many educational systems, with differing cultures and traditions of teaching and learning, and professional development.

Mathematical knowledge for teaching and levels of teacher activity: towards a coordinated model

There are many theoretical frameworks which particularize the knowledge and practices required to teach mathematics. While there is agreement within the research literature that both content knowledge and pedagogical content knowledge are requirements in the teaching of mathematics (Hill, 2010; Krauss et al., 2008; Rowland, Huckstep, & Thwaites, 2005; Schoenfeld, 2011; Speer et al., 2015), there is, at present, a separation between the cognitive and situated models of teacher knowledge and a divide between the Anglo-American and European frameworks (Depaepe et al., 2013; Rowland & Ruthven, 2011). In an attempt to encompass the knowledge incorporated by mathematics teachers during their participation in lesson study, we propose an analytical model of the knowledge required to teach mathematics, commingling both of these traditions, by combining the frameworks of Mathematical Knowledge for Teaching (Ball et al., 2008) and Levels of Teacher Activity (Margolinas et al., 2005). We refer to “combining” these theoretical frameworks in the sense of Prediger et al. (2008), in order to gain further multi-faceted insight into the knowledge incorporated by teachers in their participation in lesson study.

Mathematical knowledge for teaching

Ball and her colleagues (2008) introduced a framework of Mathematical Knowledge for Teaching (MKT), developed as a practice-based theory of the knowledge required “to carry out the work of teaching mathematics” (p. 395). This model built on Shulman’s theoretical suggestion of PCK as a specific type of knowledge unique to teachers and distinguished it from subject matter or content knowledge. In this model, Ball and her colleagues highlighted particular categories of knowledge within the PCK and subject matter delineations (see Figure 2). For example, knowledge of content and students (KCS), a sub-domain of PCK, is presented in this model as the "knowledge that combines knowing about students and knowing about mathematics" (p. 401), while knowledge of content and teaching (KCT) refers to knowing how to sequence, represent or explain the mathematics being taught. Specialized content knowledge (SCK), categorized as a distinct category of subject matter knowledge, represents knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas to learners.

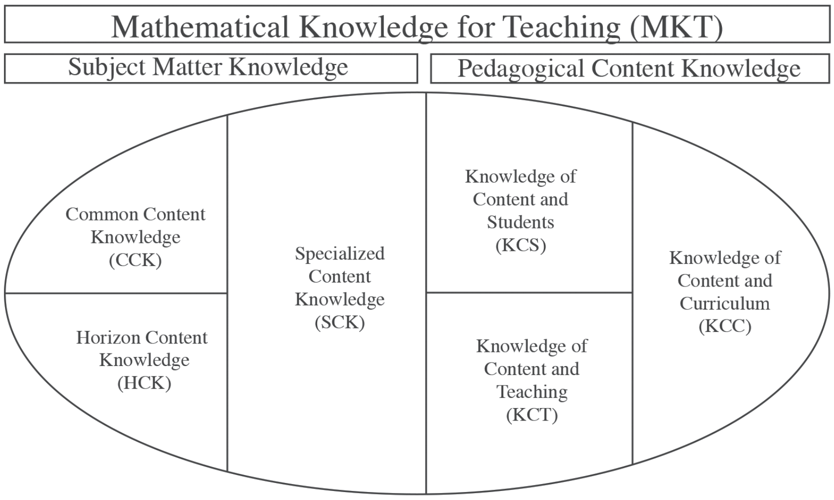

Figure 2. Mathematical Knowledge for Teaching (Ball et al., 2008, p. 403)

The categorization of these types of knowledge and the quantitative measure of their influence on student learning of mathematics have been widely studied internationally (e.g. Hill, 2010; Speer et al., 2015) and, in their review of the conceptualizing and evidencing of PCK in mathematics education research, Depaepe et al. (2013) noted this model as "probably the most influential re-conceptualizations of teacher PCK within mathematics education" (p. 13).

However, Ball and her collaborators acknowledge that these categorizations of teacher knowledge can be interpreted as static and distinct (Ball et al., 2008). This cognitive perspective of mathematics teacher knowledge, where the concept of classroom practice can be conceived of as static and fixed, does not often focus on describing how teacher knowledge influences teaching and learning (Hodgen, 2011). Neither does it include reference to a teacher's beliefs or orientations towards the subject of mathematics and their underpinning philosophy on the teaching and learning of mathematics (Petrou & Goulding, 2011).

Steinbring (1998) and Margolinas (2004) suggest that in Shulman's proposed framework of teacher knowledge (1986), on which the MKT framework is modelled, fixed categories of teacher knowledge are "not a good model for teachers' activity, which is more complicated" (Margolinas et al., 2005, p. 207). Investigating teacher knowledge and the impact such tacit knowledge may have on teaching and learning requires further exploration, since such knowledge may not be easily identified nor readily measured (e.g. Fauskanger, 2015). As stated by Davis and Renert (2013) understanding the relationship between teacher knowledge and student learning "will require more fine-grained analyses than large-scale assessments" (p. 264) to capture the sophisticated and enactive mix of knowledge utilised by mathematics teachers. The theory of didactical situations (Brousseau, 1997) provides researchers with such a tool to conduct qualitative, fine-grained and mobile analyses and is described further below.

Levels of teacher activity

In its initial stages in the 1970s, the theory of didactical situations (Brousseau, 1997) first modelled a learning situation where the teacher was largely absent from the analysis of student learning (Bloch, 2005). However, from the 1990s, the importance of the teacher's role became increasingly evident in the study and theorization of ordinary classroom situations (Bloch, 1999; Dorier, 2012; Roditi, 2011). This provided a platform to introduce a situated theory, embedded in the context of the classroom, analyzing the various levels of practices, skills and knowledge required of mathematics teachers.

The concept of *milieu* is central to the theory of didactical situations. The *milieu* is defined by "all of the pertinent features of the student's surroundings, including the space, the teacher, the materials and the presence or absence of other students" (Warfield, 2014, p. 66). Based on Brousseau's theory (1997), Margolinas (2002) developed a model of the mathematics teacher's milieu (see Figure 3) to describe a teacher's activity, both in and outside of the classroom. This model was designed to take into account the complexity of teachers' actions and capture the broad range of activities contained in teaching and learning (Margolinas et al., 2005).

+3	*Values and conceptions about learning and teaching*
+2	*The global didactic project*
+1	*The local didactic project*
0	*Didactic action*
- 1	*Observation of pupils' activity*

Figure 3. Levels of teacher activity (Margolinas et al., 2005, p. 207)

The model is not intended as a linear interpretation of teachers' work, but rather identifies the multidimensional tensions involved in classroom practices (Margolinas et al., 2005). At every level, the teacher not only has to deal with the current, most prescient, level of activity, but also with the levels directly before and after and, in some instances, with levels extending beyond. A more developed version of this model is used, for example, by Clivaz (2017) and shows how a classroom situation (*the didactic situation*, *level 0*) can be analysed both from the student's and the teacher's points of view. The teacher's point of view can be related to his/her reflections at different levels of generality. Observing students' work (including student talk) relates to a more refined focus of the teacher and individual students and, hence, relates to level -1. Planning the local didactic project (about the lesson) relates to the content of the lesson as relative to their students and, hence, is placed at level +1. At level +2, the teacher considers the didactic project in a more global sense (e.g., teaching a particular element of a topic as one lesson in a series of lessons). While at level +3, the teacher considers their beliefs on the teaching and learning of mathematics, which can be related to how the global and local projects may be constructed and to how they will engage with individual students. At every level of environment (or milieu) the teacher must consider all that is occurring at the current level, as well as those levels that are directly above and below.

As a situated model of teacher knowledge, based within the context of teaching and learning practices, Margolinas and colleagues (2005) propose a model delineating the multi-level knowledge required of teachers during varied stages of teaching, from the over-arching pedagogical values underpinning a lesson, to the didactic action within the classroom. However, while this model incorporates teacher values as well as an acknowledgement of the pedagogical skills required to notice and interpret student thinking, it does not make explicit association as to how a teacher's specific content or pedagogical content knowledge may be encompassed in such activities.

A proposed theoretical framework

Domains of MKT (Ball et al., 2008) have been shown to be incorporated and developed through teachers', both pre-service and in-service, participation in lesson study (Leavy & Hourigan, 2016; Ni Shuilleabhain, 2016; Tepylo & Moss, 2011). However, considering the multitude of knowledge and practices incorporated within each phase of lesson study – studying the curriculum, planning, conducting, and reflecting on a mathematics lesson – the MKT framework does not capture the incorporation of teachers' beliefs nor the considerations involved in structuring content for a research lesson. Ni Shuilleabhain (2015) utilized the MKT framework to investigate teacher learning in lesson study and, in an attempt to capture the knowledge incorporated by teachers in their planning and reflection conversations, combined this with the idea of a critical lens relevant to student thinking (as suggested by Fernandez et al., 2003) as an additional layer of analysis of teacher learning in lesson study. This concept of a 'student lens' relates to the PCK a teacher utilizes in seeing mathematics "through the eyes of their students"

(Fernandez et al., 2003, p. 179), but is separate to an action of the teacher noticing students' mathematical work in teaching (Jacobs, Lamb, & Philipp, 2010). This layered model relates to the work by Clivaz (2014, 2017), who used the situated activity model (Margolinas et al., 2005) to observe teacher classroom practice and aligned it with the cognitive framework of MKT in an effort to detail both the mathematical knowledge for teaching and the mathematical knowledge in teaching. However, no work, as yet, has combined these frameworks in detailing the knowledge incorporated by mathematics teachers in their participation in lesson study.

Recognizing that all theoretical approaches have their limitations as lenses for empirical phenomena, we propose a combination (Prediger et al., 2008) of these two existing frameworks of Mathematical Knowledge for Teaching (Ball et al., 2008) and Levels of Teacher Activity (Margolinas et al., 2005) as a tool which can be used to detail and analyze mathematics teachers' knowledge in various phases of planning, conducting and reflecting on teaching in lesson study (see Figure 4). The conditions and the contributions of this combination of frameworks are discussed by Clivaz (2015a, 2017). Appreciating teacher knowledge as "vast, intricate and evolving" (Davis & Renert, 2013), we propose this combined framework in an attempt to deepen our insight into the knowledge incorporated by mathematics teachers during each phase of a lesson study cycle, in the broad and complex domain of teaching and learning.

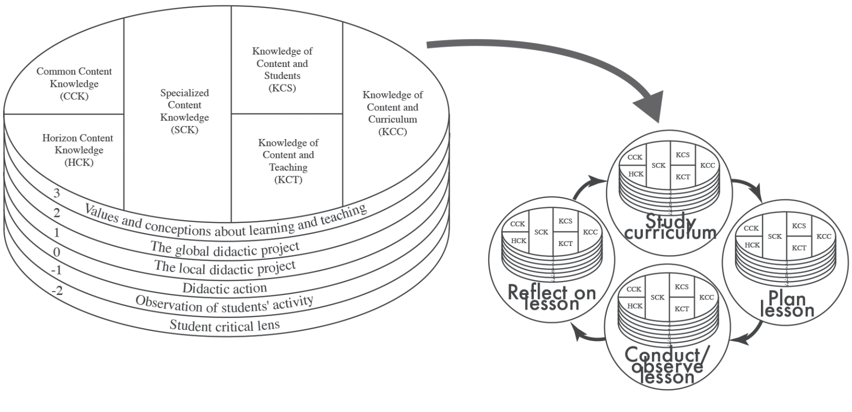

Figure 4. MKT and levels of teacher activity in a cycle of lesson study

In addition to the five layers included in Margolinas et al.'s (2005) model (see Figure 3), and based on previous analysis of teacher knowledge in lesson study (Ni Shuilleabhain, 2015), in this framework we suggest an additional level as relevant to teachers' activities in relation to specific student thinking (see Figure 4). At this level, labelled as the "student lens" and concurrent with other research on lesson study (Fernandez et al., 2003), we include teachers' consideration of mathematics "through the eyes of their students" (p. 179) where mathematical content is explicitly considered from the students' perspective. This is particularly relevant to lesson study, where teachers often engage in conversations as if they themselves were the students (Fujii, in press) and think about the mathematics from a learner as opposed to an educator perspective.

In this combined framework, the levels of teacher activity differentiate various elements of MKT drawn on during different phases of lesson study. For example, a teacher's values and conceptions about teaching and learning mathematics (*level +3*) during the planning phase (phase 1) can impact how they wish to sequence mathematical content (KCT). Similarly, the incorporation of a teacher's KCS in his/her reflection on a lesson (phase 4), can be interpreted at the local didactic level (*level +1*) or from his/her observation of students' activities (*level -1*). It is intended that this model will link the specific mathematical knowledge required of teachers with their professional activities.

As a demonstration of this model, we utilize our proposed framework to categorize and detail the levels and types of knowledge utilized and incorporated by teachers in their planning and reflection of a research lesson. In this work, we consider teachers' conversations as central to their learning (Dudley, 2013) and consider teacher knowledge as that expressed through their collaborative lesson study conversations (Robutti et al., 2016).

Methodology

Our framework is applied to two case studies, one from the Republic of Ireland and the other from Switzerland. While it has been noted that it may be difficult to introduce lesson study in varied cultures and contexts around the world (Stigler & Hiebert, 2016), we conducted this double case study in two different local contexts to acknowledge both the contextual roots of the lesson study processes and the universal characteristics of lesson study. Building on qualitative data generated through audio/video recordings of teacher conversations during one cycle of lesson study, we analyzed teachers' participation in each case study over each phase of the cycle, as articulated in their collaborative planning and reflection conversations. While these data were not originally intended for collaborative analysis, the similar format of these separate studies (Clivaz, 2016; Ni Shuilleabhain, 2015), in addition to the common theoretical approaches, has allowed us to combine these data for the purposes of this paper.

In the case study from the Republic of Ireland, five post-primary teachers, new to lesson study, and one facilitator (first author of this paper) participated in the research. These five teachers mostly taught the junior post-primary (or middle school) year groups and, as typical of post-primary mathematics teaching in the Republic of Ireland, two of these teachers reported themselves as out-of-field (i.e., not recognized as qualified to teach mathematics) (Ríordáin & Hannigan, 2011). These teachers undertook four cycles of lesson study over the course of one academic year (2012-2013), with meetings held on average every week during the school year (Ni Shuilleabhain & Seery, 2017). In the case study from Switzerland, eight primary generalist teachers (grades 3-4), new to lesson study, and two facilitators (one specialist in teaching and learning and the other a specialist in mathematics didactic, second author of this paper) participated in the research which occurred over two academic years. Four cycles of lesson study were undertaken in this time, with a meeting held, on average, every two weeks during the school year (Clivaz, 2016).

Based on the assertion that learning occurs through raised awareness generated through teacher conversations (Sakonidis & Potari, 2014), analysis was conducted on the transcribed teacher conversations from each case study. Utilizing a detailed coding matrix based on the proposed theoretical framework (see Appendix), analysis was undertaken on both case studies in parallel using the qualitative software programme NVivo. Each unit of teachers' conversations relevant to the research lesson was coded based on each of the three frameworks: MKT, levels of teacher activity and phases of the lesson study cycle. Codes and sub-codes were revised during the collaborative analysis process and a strong inter-rater reliability was established in coding the separate case studies.

In the following sections, we detail episodes of teachers' lesson study conversations as evidence of the separate levels and features of teacher knowledge incorporated in one cycle. We also include a quantitative analysis of the two data sets, demonstrating the knowledge most articulated and shared by teachers in their participation in the lesson study cycle.

Analysis of mathematics teacher knowledge in lesson study

Case 1: Multiplying Fractions

Our first example is from the Irish case study, where teachers planned a lesson on multiplication of fractions for students in their first year of post-primary school. Teachers chose to focus on the topic of fraction multiplication since they had identified common errors in students' thinking and representations of such calculations throughout students' post-primary learning. Teachers noted that while their students were often comfortable with the procedural rule of fraction multiplication, this instrumental knowledge (Skemp, 1976) did not commonly correspond to representing fraction multiples or to modelling fraction multiplication sums (Tsankova & Pjanic, 2009). Teachers wanted to support students in recognizing and realizing different forms of representing fraction multiplication, including visual models which would demonstrate a calculation (Son, 2012).

During their first planning meeting (*study curriculum* phase), Kate, a teacher who taught both senior and junior post-primary class groups, articulated an observed pattern of student error (*level -1*), while reflecting on her previous experiences of utilizing fraction multiplication during lessons.

> Kate: Like two sevenths. They know what two sevenths is – it's two over seven – but do they know what two divided by seven is? Why are they the same thing? […] We lose the division symbol. I never, ever, use the divided-by symbol the line-dot-dot. If I'm

doing distance, speed, and time formula I'll just say, you know, it's 100 over 40.

Michael: […] It's there, when you draw it. It's all over [the line].

Kate: It's 100 over 40, that's division. And they're [the students are] like, "well, it's not divided by anything, like". That is divided by!

Reflecting on her use of mathematical symbols and language in teaching fractions *(SCK)*, Kate detailed this common student misconception (Lewis & Perry, 2017) when utilizing fractions in mathematical calculations *(KCS)*. Here, operating at the level of the global didactic project (*level +2*), teachers discussed the teaching and learning of fractions at post-primary level – conveying and sharing their experiences of teaching fractions. Through this collaborative process, teachers made their implicit knowledge of teaching this topic explicit (Fujii, 2016) and built on each other's pedagogical proficiencies and learning encounters in this phase of the lesson study cycle (Ni Shuilleabhain & Seery, 2017).

Throughout this initial phase of the cycle (*phase 1*), teachers identified other patterns of student errors and common student misconceptions (*KCS* at *level -1*). For example, teachers who taught senior post-primary classes also noted that students rarely recognized as equivalent to when working on geometry problems. Due to the articulation of such issues across the vertical sequence of student learning at post-primary level (Suh & Seshaiyer, 2015), they decided to address these common errors by introducing junior post-primary students to different representations of fraction multiplication, incorporating this shared KCS into their local didactic project (*level +1*).

In planning the lesson (*phase 2*), teachers focused on their use of mathematical language when multiplying fractions (*SCK*) (Thompson & Rubenstein, 2000). Discussing the language commonly used, teachers began to unpick a common student misconception (*KCS*) that multiplication should always result in an increased value (Kerslake, 1986). Building on her years of experience teaching and tutoring mathematics, Nora reflected on her observation of pupils' activity (*level -1*) and saw the potential difficulty through the eyes of a student who she identified as having lower mathematical ability (*level -2*). She highlighted the relevance of the mathematical language linked to students' understanding of multiplication of fractions to her colleagues:

Kate: [They] forget that "of" and "multiply" are the same thing.

Nora: Yeah, but that again contradicts most people's understanding of language.

Lisa: Because they think in multiplication the answer should be a bigger than.

Nora: But, when you are telling them "of" means multiply – to my mind "of" suggests parts, suggests vision and "of" meaning

> multiply, for a lot of kids, is a big step. […] A quarter by a half, I mean, it totally contradicts every kid's instinct. You are talking about multiplying and, until then, everything gets bigger. […] "Of", to most of them, but certainly the weak students I deal with, it means part of, it means divide.

This detailing of language led the group to develop a task which compared multiplication by a fraction with multiplication by a whole number. Through their detailed conversations in planning the lesson, unpacking the mathematics (*CCK* and *SCK* at *level +1*) and articulating common student misconceptions (*KCS* at *level -1*), teachers identified the issue of 'multiplier' and 'multiplicand' with reference to multiplication tasks, content which is not traditionally included in Irish textbooks or teacher materials. Their collaborative planning prompted the group to undertake independent research in identifying the order of multiplication (e.g. Son and Senk (2010) and this element of *kyozai kenkyu* (researching teaching and learning materials and resources related to the topic) (Takahashi & McDougal, 2016) at the global and local didactic project (*levels +1 to +2*), and led to a development of teachers' SCK introducing new mathematical terminology previously unfamiliar to them. Teachers also began to build on their beliefs that student learning could be supported through their articulation of their mathematical thinking (*level +3*) (see Ni Shuilleabhain and Seery, 2017). Based on these collaborative conversations, teachers developed and modified tasks for students which highlighted the values of the multiplier and multiplicand (Taber, 2007), incorporated visual models of fraction multiplication (Tsankova & Pjanic, 2009), and encouraged students to make sense of the mathematical content of the task through working collaboratively with their peers in small groups.

In the research lesson, a number of student groups correctly provided different representations of a fraction multiplication, but others were not as successful (see Figures 5 and 6).

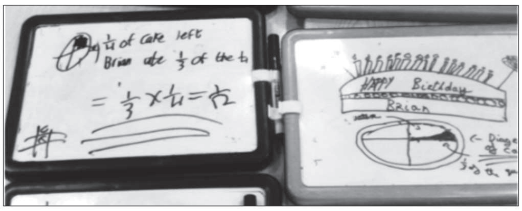

Figure 5. Student work of correct multiplication representation

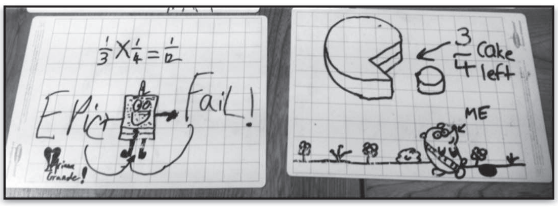

Figure 6. Student work of incorrect multiplication representation

In their reflection of the research lesson (*phase 4*), teachers were unconvinced about their sequencing of the mathematical content (*KCT*), since many students they observed were unsuccessful in demonstrating a sense of measure of a fraction (*level -1*). Students were also unsuccessful in multiplying a whole number by a fraction.

> Lisa: They put all the fractions in relation to a half, so when they were asked to do two thirds they didn't take the number line and say "well, there's a third, a third and so there's two thirds". They just said, "it's greater than a half" […] They weren't doing fifths, they weren't doing tenths, they weren't doing elevenths, they were just doing it in relation to a half. I just didn't think they got the sense that the number line was to be divided up into so many more parts. […] Then the example of two multiplied by three over two and some of them went six over four, they multiplied.

In addition, teachers who had observed the lesson suggested more emphasis should have been made on the order of the multiplier and multiplicand.

> Lisa: When the fraction was first and it was multiplied by a number greater than one, they seemed to come to some confusion whether it was going to be less than, greater than or equal to [the multiplier].

Participating in lesson study provided teachers with opportunity to plan a research lesson which greatly varied from the textbook (Ni Shuilleabhain & Seery, 2017) and which developed their SCK of multiplying fractions. Through their collaborative lesson study work, teachers realized the extent of content and concepts required of students in building a relational understanding of both the processes and representations of multiplying fractions. Across each of the phases of the lesson study cycle, they began to articulate and consider the mathematical content from the perspective of the student, building on their observations of the research lessons (*SCK* at *level -1*) and reflecting on common student errors observed in prior teaching (*KCS* at *levels -1* and *-2*). Across this cycle, teachers incorporated all elements of their MKT over various levels of teacher activity, as further described below.

Case 2: Integers and Place Value

In the Swiss case study, we present the analysis of a lesson study cycle where teachers chose to focus on the topic of integers and place value. Teachers chose this topic due to their observations of students' difficulties in working with large whole numbers. In the first session, teachers discussed the particular problems their students had with counting in double digits (*KCS* at *level -1*):

> Océane: The counting through to the next ten.

Caroline: But each time they have to count through to (tens, hundreds…)

Caroline: It's… that we have no more to write here! We have to use the digits which already exist. So, we count through to come back to one. In fact, yes, it is the abacus, in fact, we need to move by one each time we arrive at a nine at the end. We need to move by one.

Océane: We exchange one packet of ten.

In this passage, during the study curriculum phase (*phase 1*), teachers spoke about the global didactic project (*level +2*) and unpacked the mathematical knowledge required to count in base ten (*SCK*), referencing both the place and value aspects of the numbers. To further address this knowledge, the facilitators suggested working on examples of student mistakes. Teachers and facilitators proposed mistakes like:

5 hundreds + 12 tens + 3 units = 515

This work prompted teachers to do the task as if they themselves were students – placing them at the level of the student lens (*level -2*) and unpacking further the SCK about place value. This allowed teachers to realize the potential of mathematical difficulties for students (*KCS*) and, by further studying curriculum materials (*kyozai kenkyu*), teachers had opportunity to clarify this aspect of teaching (*KCT*) for the research lesson.

Following on from these planning discussions, the group chose to include a task in the form of a board game for students, which involved the exchange of "1 hundred", "1 ten" and "1 unit" cards (Batteau & Clivaz, 2016).

During the post lesson discussion, based on observations of students' work (*level -1*), teachers agreed that the task should be modified to allow students to practice the exchange of values of units, tens and hundreds with the cards. This revised task was included in a new version of the research lesson, taught by another member of the group to a different group of students. At the beginning of the game, a student, Julie, arrived on the square "give 35". She had three cards of "1 unit", three cards of "1 ten" and four cards of "1 hundred". In order to get three cards of "1 ten" and two cards of "1 unit", Julie wanted to exchange two "1 hundred" cards. The teacher, Edith, wanted to explain to Julie that two "1 hundred" cards were worth more than these three cards of "1 ten" and two cards of "1 unit":

Edith: So, two hundreds - that's how many?

Julie: Two hundred.

Edith: That's two hundreds. If you tell me: "I want three tens and two units." Three tens, how many is that?

Julie: Thirty.

Edith: You told me three tens makes thirty. And what about two units?

Julie: Two.

Edith: If you put the thirty and the two together? How many is that?

Julie: Thirty-two.

Edith: So, you swap two-hundred for thirty-two! You're very generous!

In this passage situated during the *conduct lesson* phase, at level 0 (*didactic action*), the teacher converted all cards into numbers to compare them, instead of doing direct exchanges. The student Julie followed the teacher without expressing her own way of reasoning (which can be observed in another passage and demonstrates a 'direct exchange' way of thinking). In this case, we categorize the MKT in two ways: first as a KCS, where Edith did not notice or interpret Julie's mathematical thinking or strategies, and second as a SCK, related to the unpacking of mathematical knowledge, as detailed in the following excerpt where Edith had to explain that one hundred is the same as ten tens. Here, again, the conducting teacher's strategy is to convert to units – which requires students to already understand place value. This argument can be summarized as follows:

1 hundred = 100 units
and 10 tens = 100 units
therefore, 1 hundred = 10 tens

In their lesson study report, the teachers reflected on this strategy as a way of hiding the exchange between hundreds and tens, instead of focusing on it.

> Often exchanges are not really carried out and we go through the number. For example, when asked to exchange 12 hundreds into tens, many students (and adults) will go through the number 1200, namely 1200 units, to say that that 1200 is 120 tens, without being able to make a direct exchange from hundred to tens. Teachers also often explain this exchange in this way. In this case, we are in a type of vicious circle, since it means that it is necessary to have understood number system to understand the grouping/ungrouping in the place value system.

This SCK was evident in the post lesson discussion (*reflect on lesson* phase, *level 0*) and, in the above extract, in the final lesson report (*reflect on lesson* phase, *level +2*), where observations and analysis of the group were generalized and decontextualized from the particular lesson to the level of a global didactic project.

The final example of this knowledge was found at the end of the *reflect on the lesson* phase. After discussing the lesson and the mathematical difficulty of directly converting hundreds into tens, Valentine (a teacher with over 30 years of teaching experience) realized she had observed a similar difficulty in her own students in this topic, outside of the lesson study group. As a result of their collaborative reflection conversations, she began to realize that her students' errors were likely due to her use of only one strategy in teaching this topic:

> Valentine: But, I've got a question. For example, in nine-hundred-sixty-three - how many tens are there? Ninety-six. But my students, they learned a trick - they write the number 963 and just go to the tens digit and write what is left: 96. I'm convinced they just use this trick. I probably didn't know how to explain that to them! Myself… I always convert in money! You will have nine hundred and sixty-three one-franc coins. If you need to only have ten-francs notes… then you will have ninety-six ten-francs notes.

This conversation incorporated teacher's CCK and her KCS in interpreting students' responses and is situated at level -1 (*observation of pupils' activity*).

Quantitative analysis of the coded data

From our collaborative analysis of the knowledge incorporated by teachers in a cycle of lesson study, we present tabulated findings of our coded data according to our proposed theoretical framework of mathematics teacher knowledge. Our analysis provides evidence that over one cycle of lesson study, during different phases of lesson study, all elements of mathematical knowledge for teaching as proposed by Ball et al. (2008) are encompassed in the work of planning, conducting and reflecting on a research lesson study. In addition, our data also evidences that all levels of teacher activity (Margolinas et al., 2005) are accessed at some point during a lesson study cycle. For example, while a focus of a lesson study cycle is the research lesson (*level 0*) and its preparation (*level +1,* often incorporating *CCK*, *HCK*, *KCC*), teachers are also provided with opportunities to incorporate their knowledge of students' work and thinking (drawing upon *KCS*, *KCT* & *SCK*) in sharing their noticing and interpretation of student work (*level -1*) and also seeing the mathematics through the eyes of the student (*level -2*).

Below we detail two specific examples of our case study analyses. Taking phase 2 of the Irish case study (*planning the research lesson*), each of the coded items of teachers' conversations for that phase are recorded according to their categorization of MKT and levels of teacher activity. For example, teachers utilized their SCK of the global didactic project of teaching and learning fractions (*level +2*) (4.5% of all conversations in phase 2), while also incorporating their KCS at five different levels of teacher activity (totalling 26.4% of conversations in phase 2) in critically reflecting on student work and student thinking in relation to this topic. In addition, teachers' knowledge of KCT was incorporated in their conversations, particularly in planning and sequencing content related to the local didactic project (*level +1*) (25.5% of conversations in phase 2). During this phase of lesson study, all levels of teacher activity were accessed and all elements of MKT, emphasizing PCK and SCK, were drawn on in their planning work (see Figure 7).

	CCK	HCK	SCK	KCS	KCT	KCC
+3	0.0	0.0	0.2	1.6	2.3	1.6
+2	0.0	1.2	4.5	5.4	4.2	6.3
+1	4.9	0.7	12.0	9.6	25.5	1.4
0	0.0	0.0	0.9	0.7	0.9	0.0
-1	0.5	0.2	2.1	8.2	3.5	0.0
-2	0.0	0.0	0.5	0.9	0.0	0.0

Figure 7. Phase 2: percentage of knowledge categories – Case study from Ireland

Taking the example of the entire coded data from the full cycle of lesson study in the Swiss case study, all elements of MKT and levels of teacher activity were incorporated during teachers' collaborative lesson study work (Figure 8). In this case, for example, when teachers incorporated their KCT in the topic of number and place value, these elements of teachers' knowledge were most often articulated at the level of the local didactic content (*level +1*) (20.3% of conversation) where teachers detailed, planned, and sequenced the mathematical elements of the research lesson and at the level of students' thinking (*level -1*) (14.4%) where teachers reflected and built upon their experiences of noticing and interpreting student thinking in this topic.

	CCK	HCK	SCK	KCS	KCT	KCC
+3	0.0	0.2	1.1	0.1	1.0	0.5
+2	0.1	0.5	8.3	3.6	1.9	3.3
+1	5.7	0.0	7.7	0.7	20.3	6.6
0	0.1	0.0	2.0	1.4	10.2	0.3
-1	0.0	0.0	0.4	6.6	14.4	0.9
-2	0.9	0.0	0.1	0.8	0.5	0.0

Figure 8. Percentage of knowledge categories coded over full lesson study cycle - Case study from Switzerland

Utilizing our proposed theoretical framework to detail the types and levels of knowledge incorporated by mathematic teachers in their participation of lesson study, we have analyzed the knowledge incorporated and drawn on by teachers in their participation in one cycle of lesson study in two case studies. As demonstrated in figures 7 and 8, content and pedagogical content knowledge are core to the knowledge incorporated by teachers in their participation in lesson study. In addition, all levels of teacher activity, from values on teaching and learning to thinking about the content through the eyes of the student, are incorporated in teachers' collaborative conversations over each phase of the lesson study cycle.

Discussion and conclusion

Categorizing the knowledge required to teach mathematics is complex. It is equally challenging to encompass the knowledge required of and incorporated by mathematics teachers involved in the collaborative work of lesson study. In this paper, we have proposed an extended theoretical framework in an attempt to provide a fine-grained analysis of the knowledge utilized and incorporated by mathematics teachers in their participation in lesson study. In analyzing the knowledge incorporated by mathematics teachers in their participation in lesson study, Ball et al.'s (2008) MKT framework provides a detailed categorization of the content-specific knowledge included at each phase of the lesson study cycle (e.g. Leavy & Hourigan, 2016). In parallel, Margolinas et al.'s (2005) framework of the levels of teacher activity provides means to describe a teacher's way of using his/her knowledge at various levels, from considering students' thinking to situating student learning in the broader educational system. However, a single theoretical approach may alone not fully describe an empirical phenomenon (Prediger et al., 2008) and teachers' lesson study conversations include ways of knowing about teaching and learning which can be thought of both from a cognitive perspective (e.g. Ni Shuilleabhain , 2016) and from a situated perspective (e.g. Murata, Bofferding, Pothen, Taylor, and Wischnia, 2012). In an attempt to encapsulate and capture teachers' knowledge articulated in their lesson study conversations, we have proposed a combination of two frameworks from both of these perspectives. This combination of frameworks, resulting in a matrix of alternative perspectives of teacher knowledge, is employed in an attempt to define and detail the knowledge incorporated by mathematics teachers in their participation in lesson study.

Based on case study data generated through mathematics teachers' participation in lesson study in two cases, one in Ireland and the other in Switzerland, our analysis has demonstrated that, in planning and reflecting on research lessons, all elements of MKT (Ball et al., 2008) across all levels of teacher activity (Margolinas et al., 2005) are incorporated over the phases of a cycle of lesson study. Our findings demonstrate the strength of the lesson study model in requiring teachers to consider all levels of their pedagogical activities (Margolinas et al., 2005) through collaboratively planning, conducting, observing, and reflecting on a research lesson while, simultaneously, drawing on and sharing elements of their MKT (Ball et al., 2008) through those same actions. This framework may provide further opportunity to track the development of these forms of teacher knowledge across a teacher's participation in lesson study and further work is necessary to describe the evolution of teacher knowledge within a cycle and across iterative cycles of lesson study.

The strength of drawing on two case studies from two different countries, where educational cultures and systems differ, demonstrates lesson study as a model which has the potential to explicate and develop teacher knowledge, regardless of the language or system in which it is being conducted. However, our research is limited by the fact that only two case studies of singular cycles of lesson study are included here and further research is required on additional examples of lesson study in differing educational contexts.

Additional research is also required on the utility of this framework in tracking the development of a teacher's knowledge through his/her participation in lesson study. While much work is underway in constructing theoretical underpinnings of teacher learning in lesson study (e.g. Winsløw, Bahn, & Rasmussen, 2017), we hope this work will contribute to further the analysis and research of mathematics teacher learning in lesson study.

Notes

[1] Milieu is the usual translation for Brousseau's French term "milieu", but, in French, it refers not only to the sociological milieu but it is also used in biology or in Piaget's work. A more accurate translation would be "environment".

[2] The vast majority of post-primary students in Ireland continue to study mathematics throughout their post-primary education and mathematics is a compulsory matriculation subject for the majority of third level courses (Treacy & Faulkner, 2015).

[3] Traditionally, there are no teacher manuals for textbooks in Ireland.

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Appendix

The codes (in bold) for Lesson Study Phase, Mathematical Knowledge for Teaching and Level of Teacher Activity were attributed according to the following indicators (in italics):

Lesson Study Phase

Consider issues and formulate general goals

In/for student learning and development
In/for teaching
In/for teacher's professional knowledge

1 Study curriculum and formulate content specific goals

Consider learning of the topic
Identify topic of interest
Formulate goals for student learning specific to the topic
Discuss a learning trajectory related to the topic through grades

Identify/analyse specific difficulties
In teaching
In student knowledge or learning
In content

Study curriculum, standards and material
Study course of study, standards...
Study textbook, specific task, manipulative...
Link topic to other topics
Read and reference research literature

2 Plan

Select (or revise) content
Select (or revise) research lesson
Select (or revise) sequence of lessons

Consider elements of the research lesson
Long term goals
Learning objectives
Model of learning trajectory
Rationale for chosen approach

Detail the conduction of the lesson

Anticipated student thinking

Anticipate teacher's actions

Incorporating resources (tasks, material)

Data collection plan

3 Do research lesson

Conduct research lesson

Observe and collect data

4 Reflect

Use the data to illuminate

Student actions

Student learning

Teacher actions

Disciplinary content

Lesson and unit design

Reflect on curriculum

Documentation of cycle

Consolidate and carry forward learning

New questions

Reflect about other teachings of the research lessons

Reflect about other teachings of the research lessons

Mathematical Knowledge for Teaching

Common Content Knowledge (CCK)

Performing mathematical task

Use of notations and vocabulary

Determining if a solution, a definition, a representation… is correct

Horizon Knowledge (HCK)

Considering other uses of a mathematical knowledge

Specialized Content Knowledge (SCK)

Looking for patterns in student errors

Sizing up whether a nonstandard approach would work in general

Unpacking of mathematics

Understanding different interpretations of a concept/techniques appreciating the differences

Talking explicitly about how mathematical language is used

Choosing, making and using mathematical representations effectively

Explaining and justifying mathematical ideas

Analysing/building examples having mathematical characteristics

Determining if a mathematical concept or rule is a convention or a mathematical necessity

Knowledge of Content and Teaching (KCT)

Sequencing mathematical content

Selecting models, representations, examples, and procedures that support the development of mathematical understanding

Anticipating/analysing teacher's reaction to students' response or difficulties

Anticipating/analysing teacher's actions in relation to mathematical content

Sharing or comparing representations and procedures in teaching

Selecting appropriate mathematical language, analogies and metaphors

Knowledge of Content and Students (KCS)

Identifying students' knowledge or learning

Identifying students' difficulties or misconceptions

Anticipating students' mathematical responses

Noticing and interpreting the mathematical meaning associated with students' responses

Choosing an example that students will find interesting and motivating

Selecting questions and tasks that seek out the presence of misconceptions

Knowledge of Content and Curriculum (KCC)

Linking mathematical knowledge to the syllabus (maybe implicit)

Levels of Teacher Activity

3 Values and conceptions about learning and teaching

2 The global didactic project

1 The local didactic project

0 Didactic action and observation

-1 Observation of pupils' activity

-2 Student critical lens