Figural concepts in proving by contradiction

Conceitos figurativos na prova por redução ao absurdo

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Abstract. Geometrical thinking in proving by contradiction involves specific and complex processes that can be source of difficulties for students. The goal of this article is to investigate on proof by contradiction in geometry, with particular emphasis on processes related to the treatment of the geometrical figures. The analysis, carried out with the lens of the Figural Concepts and Cognitive Unity frameworks, reveals that students, in order to conclude a proof by contradiction, need to restore the rupture between figural and conceptual components, and to try to give a geometrical meaning to the contradiction. Therefore, if in a proof by contradiction the involved geometrical figures have to be rejected (having deduced a contradiction), in the indirect argumentation proposed by many students, the figures are modified so that they do not appear absurd and impossible.
Keywords: argumentation and proof; proof by contradiction; indirect argumentation; figural concepts.

Resumo. O pensamento geométrico na prova por redução ao absurdo envolve processos específicos e complexos que podem constituir uma fonte de dificuldades para os alunos. O objetivo deste artigo é investigar a prova por redução ao absurdo em geometria, com particular enfoque nos processos relacionados com o tratamento das figuras geométricas. A análise, desenvolvida com as lentes do quadro teórico dos Conceitos Figurativos e da Unidade Cognitiva, revela que os alunos necessitam de restaurar a rutura entre as componentes figurativas e as conceptuais e atribuírem um significado geométrico ao absurdo, para concluírem uma prova por redução ao absurdo. Contudo, se numa prova por redução ao absurdo as figuras geométricas envolvidas tiverem de ser rejeitadas (tendo-se obtido uma contradição), na argumentação indireta proposta por muitos alunos, as figuras são modificadas para que não pareçam absurdas nem impossíveis.
Palavras-chave: argumentação e prova; prova por redução ao absurdo; argumentação indireta; conceitos figurativos.

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Introduction

Since Euclid's time, the theoretical view of Mathematics, from the organization of knowledge to the mathematical proof as validation in a theory, has been realized, in particular, in geometry. For a long time, in school, geometry has been the field for training problem-solving, proving theorems and constructing the meaning of mathematical theory (axiom, theorem, proof, etc.). In mathematics education, many papers have been published on geometry, from many perspectives. One of the most interesting and delicate aspects in geometrical thinking is the relationship between pictures and theory. According to Duval (2006), we can access to the mathematical objects only through their representations, but in geometry the relationships between a geometrical figure, that is a mathematical object defined in a mathematical theory, and the picture, that is a figure with spatial properties, have some specific problematics.

The relationship between theory and figures is particularly intriguing in a specific but very important kind of proof – the proof by contradiction. In this case, one has to manage pictures that can appear impossible, bizarre, absurd, when one considers some theoretical properties they are supposed to represent. Therefore, our hypothesis is that geometrical thinking in proving by contradiction involves specific and complex processes that can be source of difficulties for students. The investigation on these processes is the goal of the research presented in this article.

Conceptual background

Two classical and consolidated theoretical frameworks provide us suitable theoretical tools to analyse cognitive processes involved in proving by contradiction in geometry: the Cognitive Unity framework (Boero, Garuti & Mariotti, 1996; Garuti, Boero, Lemut, & Mariotti, 1996; Pedemonte, 2002), in which differences and analogies between argumentation and proof and their epistemological and cognitive roles are analysed and clarified; and the theory of Figural Concept (Fischbein, 1993) that focuses on the conceptual and figural components and their dialectical relationship in geometrical thinking.

Argumentation and proof

Argumentation and proof have been at the core of many discussions in mathematics education, that have stimulated articles and books on this theme carried out from different perspectives that have different didactical implications (see, for example, Boero, 2007; Hanna & de Villiers, 2012; Mariotti, 2006; Stylianides, Bieda & Morselli, 2016). For the purpose of the study presented in this paper it is suitable the theoretical framework on Cognitive Unity (Garuti et al., 1996; Pedemonte, 2002), developed on epistemological and cognitive analysis of the notion of theorem, argumentation and proof. This frame focuses on analogies between processes of generation of proof and argumentation and considers a mathematical proof as a particular argumentation satisfying some logical constraints and constructed within a reference mathematical theory (Mariotti, Bartolini Bussi, Boero, Ferri, & Garuti, 1997). The word 'theorem' usually refers to a statement of a proposition that has a significant role in a mathematical theory and that has been proved in that theory. From a historical and epistemological point of view, a statement and its proof have a history. The statement has an origin, when some mathematician generated it as a conjecture and considered it reasonably true, an evolution through different re-formulations coherently to the mathematical theories and proofs that legitimized it as a theorem. Therefore, looking at the processes, a theorem consists of different elements, in particular a conjecture, initially supported by some argumentations (by analogy, empirical, by authority, etc.), then proved by a mathematical proof that makes sense within a mathematical theory.

The Cognitive Unity framework holds the importance of processes of conjecture generation and the production of argumentation supporting the conjecture as key processes for proving and for making sense to the meaning of "mathematical proof". In other words, in a didactical perspective, a proof should not be considered in itself; it is not possible to grasp the sense of a mathematical proof without linking it to a statement and the theory, i.e. the theoretical frame within which proof makes sense. With the aim of expressing this complexity, mathematical theorem is considered as a system of *statement*, *proof* and *theory* (Mariotti et al., 1997).

Moreover, argumentation is fundamental from an epistemological point of view and becomes crucial for the construction of a mathematical proof from cognitive point of view. In fact, the framework of Cognitive Unity is based on the following:

> during the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices; during the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the justifications ("arguments") produced during the construction of the statements according to a logical chain. (Garuti et al., 1996, p. 113)

The main didactical implication is the importance of considering the theorem as a unit that includes the production of the conjecture and the argumentation, through tasks asking students to explore a situation, to generate a conjecture and an argument supporting it, and finally to produce a mathematical proof. They are tasks with much educational potential to develop argumentative and proving competencies, and to develop the meaning of mathematical proof and theory.

In summary, the research in this field, in the Cognitive Unity framework, deals with the analysis of processes involved in the conjecture generation, argumentation and proof, setting up models to identify and analyse processes, to describe differences and relationships between argumentation and proof, and to provide a theoretical basis for teachers to design and conduct didactical activities.

Figures and theory

Different contributions can be found in the literature of mathematics education referring to the role of visualization in the resolution of geometry problems (see, for example, Duval, 1998; Fischbein, 1993) and within a dynamic geometry environment (see, for example, Laborde, 1998; Mariotti, 1995).

From the logical point of view, geometry has to be considered a theory without any reference to reality, but from the cognitive point of view, the importance of pictures and diagrams has been widely discussed (Giaquinto, 1992; Jones, Gutiérrez & Mariotti, 2000; Hanna & Sidoli, 2007; Mancosu, Jørgensen, & Pedersen, 2005). Deductions make sense within a theoretical context, but their meaning and their justification value often refer to representations of geometrical figures.

The fundamental work by Fischbein (1993) is based on the assumption that activities in elementary (Euclidean) geometry involve mental entities that cannot be considered either pure concepts or mere images. Geometrical reasoning involved mental entities that simultaneously possess both conceptual properties (as general propositions validated in the Euclidean theory) and figural properties (as shape, position, magnitude). Fischbein called these entities *figural concepts*. The theory of figural concepts provides us with an efficient theoretical tool to analyse cognitive processes in geometrical thinking, as the formulation of a conjecture, and the generation of argumentation and proof. Processes can be analysed in term of the interaction between images and concept. On the one hand, the productive reasoning requires that the conceptual and the figural components blend in a figural concept, on the other hand, mistakes and difficulties can be explained in terms of the rupture, conflict or incomplete fusion between the components (see, for example Mariotti, 1993; Mariotti & Fischbein, 1997).

Proof by contradiction

The relationship between figural and conceptual components can become crucial in a proof by contradiction (or "indirect proof", as sometimes it is called), where properties derived at the theoretical level may conflict with the images.

In fact, generally speaking, a proof by contradiction starts assuming the negation of the statement we have to prove. As a consequence, the mathematical objects involved in the proof have some characteristics that are absurd and strange, sometimes in an evident way. In fact, these mathematical objects are proved to be impossible in the theory. As explained by Leron (1985, p. 23):

> In indirect proofs […] something strange happens to the 'reality' of these objects. We begin the proof with a declaration that we are about to enter a false, impossible world, and all our subsequent efforts are directed towards 'destroying' this world, proving it is indeed false and impossible. We are thus involved in an act of mathematical destruction,

> not construction. Formally, we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered. What have we really proved in the end? What about the beautiful constructions we built while living for a while in this false world? Are we to discard them completely? And what about the mental reality we have temporarily created? I think this is one source of frustration, of the feeling that we have been cheated, that nothing has been really proved, that it is merely some sort of a trick – a sorcery – that has been played on us.

In proofs by contradiction in geometry, we assume the existence of a geometrical figure and we aim to prove that this figure does not exist. In particular, in a reference theory as the Euclidean geometry, from the assumption of the (impossible) figure, we deduce a chain of propositions until a contradiction arises, that is two contradictory propositions or a proposition contradicting an axiom or a theorem. Therefore, the existence of the figure leads to a contradiction and then the geometrical figure does not exist. The figures, that are the objects of reasoning, have been constructed starting on the negation of the statement in the reference theory. They are part of a false, absurd, world, and then they have a temporary role, which ends when the contradiction arises.

I present here two proofs by contradiction with the aim to analyse the complex elements presented above.

Example 1

In the Euclidean geometry with the Playfair's version of the Euclid's fifth postulate (*In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point*), let us consider two lines crossed by a transversal. Without this postulate it is possible to prove that if the two alternate interior angles are equal, then the lines are parallel. Taking into account this proposition, we prove the opposite statement.

> *Statement:* If a line *r* is parallel to a line *s*, then $\alpha=\beta$ (see Figure 1).
>
> *Proof:* Assume that $\alpha\neq\beta$ and suppose $\alpha<\beta$. Let $\delta=\alpha$ (see Figure 2). For a proposition previously proved, *t* is parallel to *r*. Then there are two different lines, parallel to *r* and passing through the point P; this is absurd because it contradicts the Euclid's fifth postulate. Then $\alpha=\beta$.

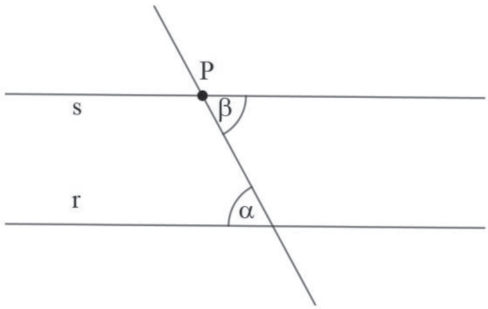

Figure 1.Two parallel lines crossed by a transversal

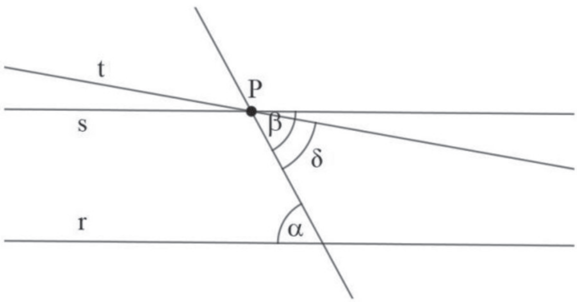

Figure 2. Two parallel lines crossed by a transversal with different alternate interior angles (impossible figure)

The proof starts with the existence of a geometrical figure with two parallel lines *r* and *s* and the relation $\alpha \neq \beta$. Within the Euclidean geometry, from the previous property it follows that there are two parallel lines to *r* through the point P that is in contradiction with the fifth postulate. Then, the initial figure does not exist.

I underline two issues related to the figural concept represented in Figure 2 and consisting in two different parallel lines *s* and *t* through P: the dynamic relationship between the figural and the conceptual components during the proving process and its role after the contradiction.

In the process of the generation of the proof, we have to think "as if" $\alpha < \beta$ and "as if" *t* is parallel to *r*, while it is evident in the picture that these relations are not satisfied.

The conceptual component of the geometrical figure allows us to make some deduction until the contradiction but the figural component is not harmonically related: to be in symbiosis, we should consider $\alpha=\beta$ and t not parallel to r.

Moreover, when a contradiction arises, we have proved that the figure does not exist and there are no any more problems about the symbiosis between the two components.

From a cognitive point of view, these issues are particularly problematic, as we will see in the following pages.

Example 2

In the Euclidean geometry, let us consider the following statement and the proof.

> *Statement*: The angle between two angle-bisectors of a triangle cannot be right.
>
> *Proof*: Let be AD the bisector of the angle $C\widehat{A}B$ and BD the bisector of the angle $A\widehat{B}C$ and let us assume that the angle $A\widehat{D}B$ is right (Figure 3). Considering the triangle $A\widehat{B}D$, $B\widehat{A}D + A\widehat{B}D = \frac{\pi}{2}$ and then $C\widehat{A}B + A\widehat{B}C = \pi$ (where π is the flat angle). Considering the triangle ABC, $C\widehat{A}B + A\widehat{B}C + B\widehat{C}A = \pi$ and then (in a triangle any angle is different from zero) $C\widehat{A}B + A\widehat{B}C < \pi$. Therefore, we have a contradiction that is the conjunction between the proposition $C\widehat{A}B + A\widehat{B}C = \pi$ and its negation $C\widehat{A}B + A\widehat{B}C \neq \pi$.

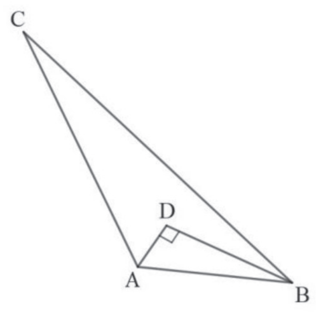

Figure 3. A (impossible) triangle with two perpendicular angle-bisectors

In this case, it could be credible that the angle-bisectors are perpendicular and the picture (Figure 3) could be a representation of a geometrical figure with this property. This can happen because of the limit of our perception (and then we cannot 'see' if the angle is exactly right) and because, in general, pictures are approximatively correct and we admit a range of errors. Differently from the previous example, we can reason on a harmonic relation between figural and conceptual components. When a contradiction is deduced, it is proved that the triangle does not exist.

Indirect argumentation

The previous analysis has revealed the complexity of proving by contradiction and makes reasonable to hypothesize that students can have many difficulties with this type of proof. In fact, studies in mathematics education, on the one hand, pointed out some specific difficulties with proof by contradiction at every school level (Antonini & Mariotti, 2007; 2008; Leron, 1985; Thompson 1996) especially in relation to the formulation and interpretation of negation, to the managing of impossible mathematical objects, to the gap between contradiction and the proved statement.

On the other hand, some studies underline that students spontaneously produce argumentations with a structure that is very similar to that of a proof by contradiction. As Freudenthal (1973) states:

> The indirect proof is a very common activity ('Peter is at home since otherwise the door would not be locked'). A child who is left to himself with a problem, starts to reason spontaneously "… if it were not so, it would happen that…". (p. 629)

Freudenthal concludes that "before the indirect proof is exhibited, it should have been experienced by the pupil" (1973, p. 629) and, along the same lines, Thompson (1996, p. 480) writes:

> If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure.

In the frame of the Cognitive Unity, we cannot refer to proof ("indirect proof") without a reference theory and I will use the term "indirect argumentation", according to Freudenthal, for an argumentation that starts from the negation of what is to be supported (for a more articulated and refined definition see Antonini, 2010) of the type "…if it were not so, it would happen that…".

We can say that indirect argumentation seems a spontaneous way of thinking, while proving by contradiction reveals some specific difficulties. From didactical point of view, in the Cognitive Unity framework, some implications arise, and in particular the importance of promoting indirect argumentation; and promoting the transition from indirect argumentation to proof by contradiction.

Studies along these lines are necessary to investigate some specific issues, interesting from theoretical point of view and for designing didactical activities, in particular with the aim to:

- set up activities to promote indirect argumentation;
- analyse argumentative processes;
- analyse the differences between indirect argumentation and proof by contradiction;
- analyse processes of transition from indirect argumentation to proof by contradiction.

In this article, I present an analysis of differences between indirect argumentation and proof by contradiction, and I focus, in particular, on the treatment of the geometrical figures in the construction of argumentation and of proof.

Methodology

The analysis presented in this article developed along both theoretical elaboration and empirical observation. The collection of the data refers to different studies (Antonini, 2008; Antonini, 2010; Antonini & Mariotti, 2007; 2008; 2010; Baccaglini-Frank, Antonini, Leung, & Mariotti, 2013, 2017; Mariotti & Antonini, 2009), it was carried out through different methodology and involved both high school students (12^{th} and 13^{th} grade) and university students (from scientific faculties as Mathematics, Physics, Biology, Pharmacy). In particular, during the interviews, carried out with pairs of students in order to favour the rise of argumentative processes, students were asked to formulate a conjecture and to prove it, in paper and pencil or in dynamic geometry environment. The mathematical lessons were recorded in order to analyse difficulties of students in understanding a proof by contradiction explained by the teacher.

Four episodes

A common way to prove an impossibility is proving by contradiction: assuming that something is possible, a contradiction is deduced; this is absurd, then the initial assumption about possibility is false. An open-ended task that promotes the formulation of a conjecture and an argumentation supporting an impossibility of some figures is expected to make indirect argumentations emerge (see Baccaglini-Frank et al., 2013, 2017).

Below, I present four episodes: three interviews to students and a part of a lesson carried out by a teacher. The first two interviews were collected during a research study concerning indirect argumentation and proof by contradiction both at university level and at high school level (see also Antonini & Mariotti, 2010; Mariotti & Antonini, 2009). The third interview was collected during a research study on maintaining dragging in dynamic geometry environments (see Baccaglini-Frank, 2010). In the interview, two students who were familiar with Euclidean geometry were asked to collaborate in order to generate a conjecture and to prove it.

The triangle becomes a line

Elenia and Francesca are two university students (first year of the Biology Faculty) involved in solving the following problem:

What can you say about the angle formed by two angle-bisectors in a triangle?

After a short exploration of the possible configurations of the triangle and the angle-bisectors, they investigate the case of orthogonality and they correctly deduce that “if the angle between the angle-bisectors is right then $2\alpha+2\beta=180$” (Figure 4). This proposition contradicts a well-known theorem about the sum of the interior angles of a triangle, then a contradiction arises. From a logical point of view, the contradiction assures that the triangle with two perpendicular angle-bisectors does not exist, then, in any triangle, the angle formed by two angle-bisectors cannot be right. Nevertheless, to accept this proof we have to share that the contradiction is a criterion of impossibility. Elenia and Francesca do not consider the contradiction as an argument to state the impossibility of the figure but they seem surprised and puzzled:

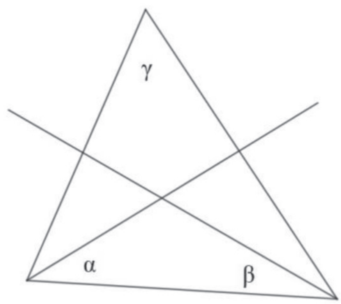

Figure 4. The Elenia and Francesca’s triangle

Elenia: … There is something wrong.

Interviewer: Where?

Elenia: In 180.

Interviewer: Why?

Elenia: Because, it [180] is the sum of all the three interior angles, isn’t it? (…)

Interviewer: Yes.

Elenia: Right.

Interviewer: And then?

Elenia: And then there is something wrong! They should be $2\alpha+2\beta+\gamma=180$.

The feeling of surprise, expressed by the claim that something is wrong, seems to come from the difficulties to conceive the figural component of the deduced properties. The contradiction appears as a non-sense, a mistake ("there is something wrong"); the students do not consider it to formulate and to argument a conjecture about the impossibility. Rather, their effort is oriented to restore the lost meaning: Elenia: "… and then it would become $\gamma = 0$".

The students try to give it a sense, drawing further propositions. The conceptual component is now a triangle with an angle equal to a null angle, and two angles forming a flat angle: there is a break between the conceptual and the figural aspect that has to be repaired:

> Interviewer: And then?
>
> Elenia: But equal to 0 means that it isn't a triangle! If not, it would be so [she joins her hands]. Can I arrange the lines in this way? No... […] And then essentially there is no triangle any more.

The conceptual aspect is interpreted in the figural aspect: $\gamma=0$ has a meaning *("means")* from a figural point of view. Then Elenia rearranges the figure coherently with the proposition $\gamma=0$, restoring the harmony between the figural and the conceptual aspect. The figure is not a triangle, two sides are overlapped and it becomes a segment.

> Interviewer: And now?
>
> Elenia: … that it cannot be 90 [degrees].
>
> Interviewer: Are you sure?
>
> Elenia: Yes. […] because, in fact, if $\gamma=0$ it means that… it is as if the triangle essentially closed on itself and then it is not even a triangle any more, it is exactly a line, that is absurd.

Now, the situation makes sense for Elenia: "The figure is not impossible; it is different from what it was before, from what we have thought it had to be". For her, the absurd is not the theoretical contradiction but the fact that "it is not even a triangle any more, it is exactly a line". A triangle with two angle-bisectors perpendicular is impossible because, in order to maintain the coherence between conceptual and figural aspects, it closes into a segment.

In summary, the figure does not have a temporary role as in a proof by contradiction; it is not destroyed (or 'discarded', as in the Leron's quote above) because of the contradiction. The figure is a dynamic entity that changes in order to make sense to the contradiction and to argument the impossibility.

The triangle becomes a quadrilateral

Paolo and Riccardo are two high school students of a *Liceo Scientifico* (grade 12) involved in the same task of the previous example (K/2 and H/2 are respectively α and β of the Elenia and Francesca's episode). They exclude the case of acute angle between angle-bisectors and they consider the case of the right angle:

> Paolo: As far as 90, it would be necessary that […] K/2=45, H/2=45 […].
>
> Interviewer: In fact, it is sufficient that […] K/2 + H/2 is 90.
>
> Riccardo: Yes, but it cannot be.
>
> Paolo: Yes, but it would mean that K+H is ... a square […]
>
> Riccardo: It surely should be a square, or a parallelogram
>
> Paolo: (K-H)/2 would mean that […] K+H is 180 degrees...
>
> Riccardo: It would be impossible. Exactly, I would have with these two angles already 180, that surely it is not a triangle. […]
>
> Riccardo: We can exclude that [the angle] is $\pi/2$ [right] because it would become a quadrilateral.

Paolo and Riccardo prove that, if the angle-bisectors are perpendicular, then K+H=180. Riccardo recognized the impossibility of this equality, but it is not considered to prove the falsity of the assumption. Their attempt to give a geometrical interpretation can be interpreted as an effort to give meaning to this proposition ("it would mean") or, in terms of the figural concepts, to restore the harmony between conceptual and figural components. Then, a new object, a quadrilateral, appears. In fact, it seems to be perceived as the same figure, the triangle that becomes a quadrilateral ("it would become") in order to satisfy the deduced property. This transformation is seen as necessary and the argument appears more convincing than the deduction of a contradiction in the conceptual component of the figure.

The triangle becomes one line or two parallel lines

Andrea and Roberto are two students attending the second year of high school (grade 10) in a northern Italian *Liceo Scientifico*. They have used *Cabri II Plus* the year before. The problem is a variant of that analysed above and they are asked to solve the problem in a dynamic geometry environment:

> *Task:* Is it possible to construct a triangle with two perpendicular exterior angle-bisectors? If so, provide steps for a construction. If not, explain why not.

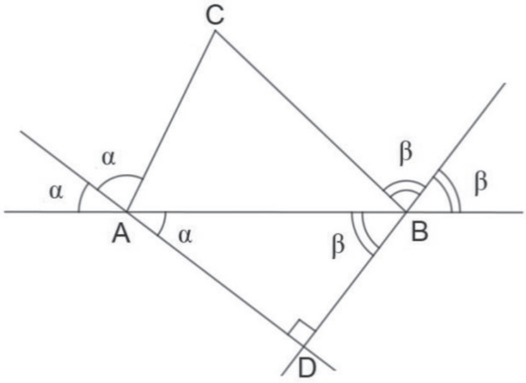

Figure 5. The Andrea and Roberto's (impossible) triangle

In the dynamic geometry environment, the two students try to construct the triangle with the requested properties and they can see that it is impossible. Finally, they argument their conjecture drawing a figure on a sheet of paper (Figure 5), and writing their argumentation:

> It is not possible, because, if D=90°, then $\alpha+\beta=90°$. Then $2\alpha+2\beta=180°$. So C∈ AB (but D is not formed because the two bisectors are parallel) or AC || CB that can only happen if A, B and C are collinear and so they do not form a triangle.

As in the previous protocols, the 'proof' (the argumentation) takes into account their necessity to give geometrical meaning to the 'strange' conclusion: $2\alpha+2\beta=180°$. This is possible, for the students, distinguishing two cases: C∈AB, and AC is parallel to C. The fact that in these cases the points "*do not form the triangle*" seems to be a convincing argument for proving the conjecture.

Two different lines become the same line

The following episode comes from some lessons I recorded in a first year of high school (grade 9) in an Italian *Liceo Scientifico* (see Antonini, 2008). The teacher presents the statement "if *r* is parallel to *s*, then $\alpha=\beta$" (Figure 2), and he proposes the proof by contradiction we analysed in a previous section:

> Suppose that $\alpha<\beta$ and let $\delta=\alpha$. For a theorem proved in the previous lesson, *t* is parallel to *r*. Then we have two lines, parallel to *r* and passing through the point P; this is false for an Euclidean axiom. Then $\alpha=\beta$.

Students are astonished, confused, and not convinced. They don't accept the proof as a convincing argument and they ask for different explanations. In particular, students do not understand the role of the line *t*, because either it doesn't exist or it is not parallel to *r*.

After many attempts to explain the structure of proof by contradiction and the role of contradiction, the teacher changes strategy and spontaneously proposes a different conclusion:

> Ok. Listen to me. For an Euclidean axiom there exist only one parallel line, then, in fact, the line *t* and the line *s* are the same line! Then the angles *b* and δ are the same angle; and, because $\delta=\alpha$, then $\beta=\alpha$.

Differently to the proof, students accept this argumentation, are convinced and they prefer it to the first one.

In summary, the teacher proposes a proof by contradiction and an indirect argumentation. In the proof by contradiction, the equality $\alpha=\beta$ is validated on the base of the contradiction between an axiom and its negation that follows from the assumption "$\alpha<\beta$". At the end of the proof, the figure does not have any geometrical sense (both *t* and *s* are parallel to *r*) because there are a deep rupture between conceptual and figural components. This figure, actually, does not exist in Euclidean geometry.

In the indirect argumentation, the figure is modified and the harmony between figural and conceptual components is restored. In the initial figure, there are three lines (*r*, *s*, and *t*), but at the end we discover that *s* and *t* are the same line. The transformed figure allows, on the one hand, to eliminate any contradiction (the Euclid's fifth postulate is not false in the transformed figure) and, on the other hand, to validate the equality "$\alpha=\beta$" and then the theorem. This argumentation appeared to students more understandable and convincible.

Discussion

The examples above are enlightening both for the analysis of processes involved in the construction of indirect argumentation and for the identification of differences between (indirect) argumentation and proof by contradiction.

It seems evident the students' requirement of generating argument about something that makes sense; in the case of geometry, this sense requires a harmony between conceptual and figural components of the figural concepts.

The contradiction is considered a mistake, an anomaly (see Antonini & Mariotti, 2010): it is not sufficient to validate a statement but it requires to be explained. In fact, the anomaly is caused by a rupture between the figural and the conceptual component, a rupture that generates an impasse. In order to overcome this impasse and complete the argument the students try to restore the unity between the two components: the anomaly, absurd from the theoretical point of view and sufficient to conclude the argumentation from the logical point of view, is reinterpreted at the figural level.

This process is well represented by the verbs "to mean" and "to become". The contradiction is not a criterion of impossibility but "it means" something and, as a consequence, the triangle "becomes" a line or a quadrilateral. The verb "to mean" expresses the students' need to give a geometrical meaning to the deduced theoretical consequences. The expressions "any more", "become", "closed on itself" refer explicitly to the dynamic status of the figure and to its transformations: the figures are not static, are not discarded because they lead to a contradiction but they are adjusted so that the anomaly can be explained, the contradiction can make sense, and the figural and the conceptual components are harmonically linked. At the end, the figure is different from the initial one and does not have the requested properties. In constructing the figure, the theoretical constraints and the search for a geometrical meaning has determined something different, then the initial figure is impossible. Once the harmony is restored the argument can be developed: new figures negate the existence of a triangle and consequently provide the missing step to validate the falsity of the assumption. Now the students feel satisfied and manage to conclude. Elenia says that "there is essentially not the triangle any more" not because its existence leads to a contradiction but because it is transformed into something different ("it is as if the triangle essentially closed on itself and then it is not even a triangle any more"); in the same way, Riccardo concludes that "we can exclude that [the angle] is $\pi/2$ [right] because it would become a quadrilateral". Therefore, the structure of the argumentation is particularly different from the structure of a proof by contradiction.

In summary, in a proof by contradiction, the geometrical figures have a temporary role that ends with the contradiction: when a contradiction arises, the figures disappear, they have to be rejected and it is stated that they don't exist. In fact, they have never existed. Differently, in the indirect argumentation we have described, the figures are modified. The transformation of the figure in indirect argumentation seems to have two important roles:

1. to overcome the problematic posed by Leron (1985, p. 323): "What about the beautiful constructions we built while living for a while in this false world? Are we to discard them completely? And what about the mental reality we have temporarily created?". The mathematical objects are transformed and adapted to the bizarre and anomalous propositions. In the case of geometry, the transformation is guided by the search of symbiosis between figural and conceptual components;
2. to prove that the original assumption is false. The impossibility of the initial figure does not follow directly from the contradiction but because of the transformation of the figure. In other words, when trying to construct the figure, it is necessarily transformed into something different. This seems to be an accepted criterion for the impossibility.

Conclusion

The above analysis has shown that students produce indirect argumentations to support some statement regarding impossibility and, at the same time, has enlightened the discussion on the differences between indirect argumentations and proofs by contradiction, by the identification of different processes of treatment of figures and of contradiction.

Regarding to the transition between indirect argumentation and proof by contradiction, there are not specific studies in mathematics education and further researches are needed. Nevertheless, the Cognitive Unity framework and the previous analysis allow to draw some implications. In particular, the differences between argumentation and proof in the case of contradiction make clear that proving by contradiction requires a specific teaching. Coherently with the Cognitive Unity framework, an efficient learning of proving by contradiction should require knowledge and awareness, namely:

1. the knowledge of the method: the assumption of the negation of the statement, the deduction within a reference theory until a contradiction, the falsification of the negation and the validation of the statement;
2. the awareness of the processes involved in the production of argumentation: the spontaneous research of a geometrical meaning, the treatment of figures with the transformation aimed at restoring the harmony between figural and conceptual components and the differences between these processes and the process of construction of proof by contradiction.

The tasks we have seen above can be completed with the request of a proof by contradiction. In particular, requiring a written proof could be an efficient tool to make students aware of the structure of their argumentation and to force them to adjust the indirect argumentation in a proof by contradiction. Metacognitive processes have a fundamental role in managing this transition that requires both the knowledge of the logical structure of proof by contradiction and the awareness of differences in the treatment of contradiction and of impossible figures, in proof and in indirect argumentation.

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