Teaching and learning Geometry: early foundations[1]

O ensino e a aprendizagem da Geometria: princípios fundamentais

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Abstract. Young children's abilities to engage in geometric thought and spatial reasoning can support their overall mathematical and cognitive development. And yet, geometry is not always addressed in early childhood curriculum and, even if included, it is not explored in ways recommended by research. In this paper, we present three studies that examined the teaching and learning of geometry with related math research and discuss curricular and instructional implications. In the first study, we examined the effects of a geometry curriculum that synthesized the visual cognition elements of the Agam program. The second study explored the impact of an early math curriculum, based on learning trajectories. The third study examined teachers' math talk and its impact on children's overall math concept acquisition. We conclude that geometry curriculum for the young child is most effective when it includes a broad array of tasks that are based on learning trajectories with varied examples and non-examples, nurtures visual cognition with progression towards analytical thinking, and integrates rich and diverse math talk.

Keywords: geometry; early childhood; visual cognition; math talk; learning trajectories; spatial reasoning.

Resumo. A capacidade das crianças para se envolverem no pensamento geométrico e no raciocínio especial pode apoiar o seu desenvolvimento global matemático e cognitivo. Contudo, a geometria nem sempre é considerada nos currículos de educação pré-escolar e, mesmo quando incluída, não é explorada de acordo com as recomendações da investigação. Neste artigo, apresentamos três estudos que analisam o ensino e a aprendizagem da geometria em relação com a investigação matemática e discutimos as suas implicações para o currículo e para o ensino. No primeiro estudo analisamos os efeitos de um currículo de geometria que sintetiza os elementos cognitivos visuais do programa Agam. O segundo estudo explora o impacto de um currículo de matemática para os primeiros anos, baseado em trajetórias de aprendizagem. O terceiro estudo analisa o discurso matemático do professor e os seus impactos na aquisição pelas crianças de conceitos matemáticos. Concluímos que o currículo de geometria para as crianças é mais eficaz quando considera um vasto conjunto de tarefas, baseado em trajetórias de aprendizagem, incluindo exemplos variados e contraexemplos, favorece a cognição visual progredindo para o pensamento analítico, e integra um discurso matemático rico e diversificado.

Palavras-chave: geometria; educação de infância; cognição visual; discurso matemático; trajetórias de aprendizagem; raciocínio espacial.

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Introduction

Geometry and spatial reasoning are not only important in and of themselves, but lay a critical mental foundation for learning other topics in mathematics as well as other subject matter areas (Clements & Battista, 1992; Olkun & Sarı, 2016; Sarama & Clements, 2009b; The Spatial Reasoning Study Group, 2015; Vallortigara, 2012; Zacharos, Antonopoulos, & Ravanis, 2011; Zorzi, Priftis, & Umiltà, 2002). Despite its importance, geometry and spatial thinking do not play a significant role in research. Here we briefly review extant research on young children's ideas and learning about shapes and then present three research projects based on that research corpus. The first study sought to gain a complete understanding of the learning processes children go through when developing visual cognition and related geometry skills. The second built on this one but incorporated a more scientific approach to curriculum development focusing especially on research-based learning trajectories. The third focused on one critical aspect of teaching early geometry, math talk. All these studies are multifaceted and thus we are presenting summaries only, rather than complete descriptions (citations are provided for those interested in details on theory, methods, and analyses). In the final section, we synthesize the findings and draw implications for teaching geometry to young children.

Teaching geometry

Another rationale for this paper is that geometry and spatial thinking do not play a significant role in the *practice* of early mathematics education (National Research Council, 2009; Ng, 2011; Sarama & Clements, 2009b; The Spatial Reasoning Study Group, 2015). Also, research tends to focus on number (Clements & Sarama, 2011; Clements, Swaminathan, Hannibal, & Sarama, 1999; Dağlı & Halat, 2016; Flevares & Schiff, 2014). For example, geometry is one of the mathematical domains least understood by teachers of young children in many countries (e.g., Clements & Sarama, 2011; Fujita & Jones, 2006; Ng, 2011; Turgut, Yenilmez, & Anapa, 2014) with negative effects (Firmender, 2011; van der Sandt, 2007). Such neglect contradicts research indicating the importance of geometry and spatial thinking to children's mathematical development (Clements & Battista, 1992; Olkun & Sarı, 2016; The Spatial Reasoning Study Group, 2015; Vallortigara, 2012; Zacharos, Antonopoulos, & Ravanis, 2011; Zorzi, Priftis, & Umiltà, 2002).

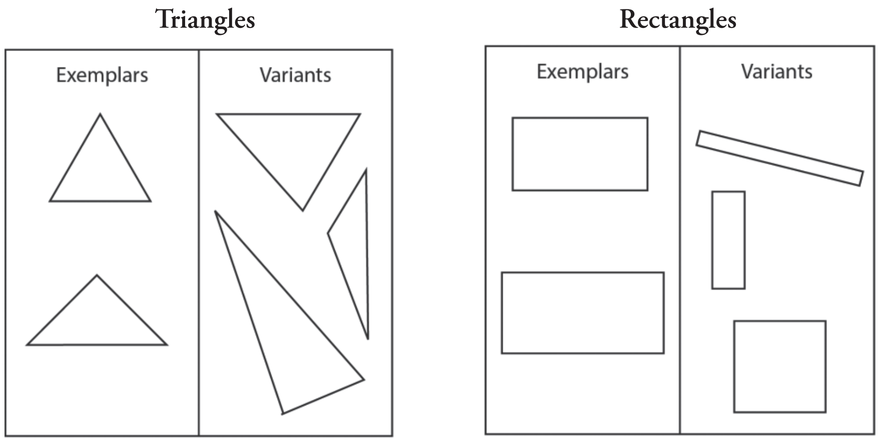
Young children's concepts of shape

Shape is a fundamental construct in cognitive development in and beyond geometry. For example, young children form artifact categories characterized by similarity among instances in shape (Jones & Smith, 2002). Children begin forming conceptual understanding of geometric figures in the early childhood years. Further, their basic concept images (Vinner & Hershkowitz, 1980) of these figures tend to stabilize by the age of 6 years (Gagatsis & Patronis, 1990), so 3- to 6-years of age may be a particularly significant period for learning about geometric figures (Clements et al., 1999; National Research Council, 2009). We focus here on specific plane figures, but agree that complete geometry experiences include a wide range of figures (especially 3-dimensional) and spatial competences.

Analysis of children's identification of figures is aided by defining specific subcategories of examples and non-examples, that is, how well children can differentiate figures that accurately represent a certain shape from those that do not. We classify geometric figures used in assessments into four subcategories (Clements & Sarama, 2014) defined in Figure 1, which provides examples for the classes of rectangles and triangles. Geometric figures are categorized as either members of the class (correct selections on shape identification items) or non-members, called distractors. Each of these categories is further categorized into those more and less representative of that category (see the definitions and examples in Figure 1). The basis for this categorization is both mathematical and psychological. Research suggests inborn or early developing geometric competences and an inborn tendency to form specific mental prototypes such as the exemplars in Figure 1 (Dillon & Spelke, 2015; Newcombe & Huttenlocher, 2000; Sarama & Clements, 2009b). Culture influences these preferences, such as the frequent presentation of common shapes (often circles, squares, rectangles, triangles) in limited prototypical forms at home and in school (Aslan & Aktas-Arnas, 2007a). Such forms are directly reflected in children's behaviors (e.g., Clements et al., 1999). In one study, 4- to 5-year olds considered rotated squares no longer the same shape or even size, 6- to 7-year-olds retained its characteristics, but lost its category and name – it was no longer a square to them, frequently named a "diamond", and only by 8- to 9 years did students achieve invariance (Vurpillot, 1976). This may reflect a systematic bias for horizontal and vertical lines and a need for perceptual learning and flexibility, but restricted experiences at home and at school exacerbate such restrictions. Research indicates that without high-quality geometry experiences, such rigid visual prototypes can rule children's thinking throughout their lives (Vinner & Hershkowitz, 1980; Zykova, 1969).

Members

Exemplars are theoretically and empirically determined prototypes of the class. Variants are other menbers of the class that differ from these examples



Nonmembers

Palpable distractors do not have an overall resemblance to members of the shape class (e.g., ovals for the triangle selection item).
Difficult distractors are highly visually similar to members but lack at least one defining attribute (e.g., a triangular shape with a "curved slide").

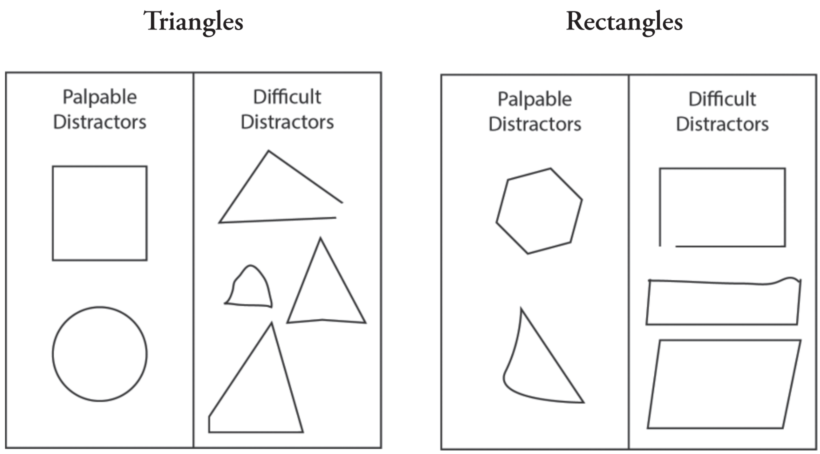


Figure 1. Examples of subcategories for the classes rectangle and triangle

Research also identifies specific prototypes of geometric shapes that children develop, with similar results reported in Singapore, Turkey, and the U.S. (Aktas-Arnas & Aslan, 2004; Aslan & Aktas-Arnas, 2007b; Clements et al., 1999; Yin, 2003). Preschoolers identify circles accurately, with only a few children choosing an ellipse and another curved shape. Most children described circles as "round", but few could offer any description. Children also identified squares fairly well, with some choosing nonsquare rhombi. Children were less accurate at recognizing triangles and rectangles (except in Turkey, Aslan & Aktas-Arnas, 2007b). However, their scores were not low; about 60% correct for triangles — an important class, as examples are visually diverse (as they would for the corresponding, but less frequently evoked, class of quadrilaterals). Both the U.S. and Singapore data revealed a phase in which children chose more triangle examples and palpable distractors, then "tightened" their criteria to omit some palpable distractors but also some examples. The children's visual prototype seems to be of an isosceles triangle with a horizontal base. Turkish children found triangles the most difficult class to classify (Aslan & Aktas-Arnas, 2007b). Asked to select rectangles, young children tended to accept "long" parallelograms or right trapezoids. Thus, children's visual prototype of a rectangle seems to be a four-sided figure with two long parallel sides and "close to" square corners. Striking across the U.S. studies is the lack of significant change from the preschool years to sixth grade (Clements et al., 1999).

In a later study, children ages 3 to 6 were asked to sort a variety of manipulative forms (Hannibal & Clements, 2010). Certain characteristics irrelevant to shape classification affected children's categorizations: skewness, aspect ratio, and, for certain situations, orientation. Orientation had the least effect. Skewness, or lack of symmetry, was more important. Many rejected triangles because "the point on top is not in the middle." Turkish children showed the same pattern (Aslan & Aktas-Arnas, 2007b). For rectangles, on the other hand, many children accepted non-right parallelograms and right trapezoids. Also important was aspect ratio. Children preferred an aspect ratio near one for triangles; that is, about the same height as width. Other forms were "too pointy" or "too flat." Children rejected both triangles and rectangles that were "too skinny" or "not wide enough." These same factors (with an additional one of size) similarly affected children's judgments in Turkey (Aslan & Aktas-Arnas, 2007b).

In another study, children of 3, 4, 5, 7, and 9 years of age, and adults, were asked to sort shapes (Satlow & Newcombe, 1998). A substantial change occurred between 4- and especially 5-years of age to 7 years, with older children relying more on rule-based definitions and less on perceptual similarity (basic concept images) than younger children. Younger children were more likely to accept palpable distractors with characteristic features and reject variants. Development regarding recognition of variants was incremental, but identification of palpable distractors showed sudden improvement. Consistent with research discussed, shapes with multiple variants, such as triangles, were more difficult. The authors state that this is not consistent with more general theories of geometric development, including Piaget's or the van Hieles', because children learn about different shapes at different times. However, the shift itself is consistent with our hierarchic interactionalism reinterpretations of these theories (Sarama & Clements, 2009b).

To contribute to such research guiding early geometry teaching and curriculum, we next summarize research on young children's learning of shape from three interventions. Because it contributed in different ways to all three, we first describe an early intervention, the Agam program.

Teaching young children geometry and spatial thinking: three studies

The Agam curriculum in kindergarten classes

The Agam program was designed to teach young children "visual literacy" (Eylon & Rosenfeld, 1990; Razel & Eylon, 1986). An artist, Agam, and collaborating educational researchers developed the Agam program to develop the visual language of children ages 3 to 7 years, composed of basic elements and their interrelationships: "Teaching visual language should be viewed as 'a basic' and should commence, within the formal school system, at a young age" (Razel & Eylon, 1990, p. 12). The activities begin by building a visual "alphabet" – foundational images from which more complex figures are built. For example, the activities introduce horizontal lines in isolation. Then, they teach relations, such as parallel lines. In the same way, teachers introduce circles, then concentric circles, and then a horizontal line intersecting a circle (for Agam, such combinations are basic). The curriculum also develops verbal language, but always following a visual introduction. Combinations involving the visual alphabet and ideas such as large, medium, and small, generate complex figures. As words combine to make sentences, the elements of the visual alphabet combine to form complex patterns, symmetric forms, and geometric units of units. The Agam approach is structured, with instruction proceeding from passive identification to active discovery, first in simple form (e.g., looking for plastic circles hidden by the teacher), then in tasks that require visual analysis (e.g., finding circles in picture books). Only then does the teacher present tasks requiring reproduction of combinations from memory. The curriculum repeats these ideas in a large number of activities featuring multiple modes of representation, such as bodily activity, group activity, and auditory perception.

The Agam program hypothesized that the visual language developed by the children was generative in nature, allowing children to extend their language to new situations and to solve new problems (Razel & Eylon, 1990). The results of using the program, especially for several consecutive years, supported this hypothesis. Children gained not only geometric and spatial visualization skills but also showed pronounced benefits in the areas of arithmetic and writing readiness (Razel & Eylon, 1990). Also, those children who participated in the curriculum instruction for two years benefited more than those who participated for 1 year or not at all.

Razel and Eylon (1990) concluded that these findings point to the educational potential of the Agam approach of systematic long-term instruction in the domain of visual cognition in early childhood. The first of our three studies was based largely on the Agam program, the other two were influenced by it.

In the first research study, we wanted to gain a complete understanding of the learning processes children go through when developing visual cognition and related geometry skills. We therefore developed a synthesized curriculum that intermingled the Agam tenets with Piagetian and van Hiele theories of learning spatial and geometry concepts. Consistent with Piaget's constructivism, our synthesized curriculum ensured that children were involved in investigations that challenged their current thinking and motivated them to explore actively. Another philosophy incorporated into this curriculum came from social constructivism that emphasizes a need for human interaction and discourse. Therefore, we ensured that the activities in our curriculum required children to work in collaborative partnerships and that there were ample opportunities for discussion and justifications. Consistent with the van Hiele levels of geometric thinking (van Hiele, 1986), we designed the activities so that the children were challenged to move beyond a visual approach towards more analytic thinking.

Description of the synthesized curriculum

The synthesized curriculum was divided into three units called Investigations. The first investigation, "Shapes," involved the children in actively thinking and working with different types and sizes of shapes with activities such as sorting and imaging. In the second investigation, "Paths, Directions and Shapes," the children started analyzing the attributes of these shapes through shape walks and shape reproduction activities. The third investigation, "Transformation," dealt with inter- and intra-shape transformations. Here the children worked on transforming one shape into another or on maintaining the properties while changing a shape's dimensions.

The control group engaged in approximately three geometry-oriented activities through the course of the study. These included shape identification (explore the environment and note objects with geometric shapes), shape composition (create geometric pictures and designs from cut-out shapes) and counting geometric parts (collect boxes and other three-dimensional shapes and count the number of faces, edges and corners).

Research Purposes

In summary, we investigated the effects of geometry instruction on the development of kindergarten children's geometric concept understanding and on transfer of this development. There were two major goals: (a) to examine the cognitive constructions of geometry concepts and visual skills that kindergarten children make in the context of instructional environments, and (b) to examine the effects of instruction of the synthesized curriculum.

Method

Participants. The participants in this study were from two kindergarten classrooms of a suburban school. The experimental group had 20 children while the control group had 18 children, all of whom were between the ages 5 and 6 years. Both kindergarten teachers had Masters degrees in elementary education and at least 10 years of teaching experience.

Implementation. The synthesized curriculum was implemented over a 7-week period with the experimental group engaging in 3 or 4 activities per week. The control group did the same number of geometric activities from a curriculum prescribed by the school district.

Data sources and analyses. The Colored Progressive Matrices (Raven, Court, & Raven, 1977) and the Goodenough Harris Draw-a-Man tests (Harris, 1963) were administered as pre- and posttests, as per the Agam research program. The former is a non-verbal measure of children's abilities to think and reason by analogy. The latter measures children's cognitive developmental levels, unaffected by language. A geometry concept test adapted from the Agam project was administered as pre- and posttests of children's geometric knowledge. A composite score derived from the two cognitive tests was used as the covariate in the ANCOVA analysis of the geometry concept test. Qualitative data included pre- and post-interviews of the experimental group coinciding with the geometry concept test, classroom observations, and case study analyses of two randomly selected children. Field notes were repeatedly analyzed by two researchers for patterns indicating progress in the children's thinking. These patterns were juxtaposed against the quantitative results to derive a comprehensive picture of the children's growth. Similarly, classroom observations, the interviews, and the test results were utilized to study the effectiveness of the curriculum.

Results

Overall, the children who participated in the synthesized curriculum evinced greater growth in their visual abilities and geometric thinking. We discuss key results and examine the impact of the synthesized curriculum on producing these results.

Shape recognition. We examined children's ability to identify shapes (circles, triangles, squares and rectangles) in tasks that varied in the degree of complexity such as identifying shapes among distractors, within the classroom environment and within a realistic picture. There was no difference in children's abilities to identify circles, with both groups demonstrating strong prior knowledge of circles and showing similar overall growth. The striking effects of the curriculum were evinced in their identification of squares and triangles. Identification of squares, in a simple context with a horizontal base, was familiar to all children. Rotated squares (those without a horizontal base) were most often confused with non-squares at the pretest. However, at the posttest, only 5% of the experimental group showed evidence of this confusion in comparison to almost 80% of the control group. Identification of triangles was originally hard for both groups of children, but the experimental group significantly outperformed the control at the posttest.

The activities that focused on developing visual thinking skills could have been the mediating factor for these results. For example, the skill of perceptual flexibility was enhanced through sorting shapes and making patterns with squares of different sizes. Children were able to perceive correctly the perceptual changes in the shapes, recognizing the square or triangle even when it was turned around. Activities such as reading books with illustrations of various shapes and discussing properties of these shapes encouraged children to analyze their environment (and the shapes therein) into simpler elements. Creating designs with attribute blocks, making shapes with their bodies and reproducing shapes with manipulatives facilitated the children's visual coding and decoding. These activities helped them visualize what the square and triangle looked like through creating and reproducing the shape rather than depending on a rigid visual image.

The qualitative inferences illuminate these findings by providing a window into children's thinking during shape recognition. Two stands of thinking were noticed. In the one, children could match the verbal name of a shape to a visual prototype, such as looking at a coin and saying that it is a circle. In the other, children could recognize the attributes of shapes but evinced difficulty in recalling its verbal name immediately but could identify it by touching a visual prototype. This was evident during the feely bag activities when children could feel the shape and its attributes but not see it. It is possible that deciphering the attributes of a hidden shape and merging these into a shape called for greater mental processing than it did to look at a shape holistically and recall its name; and it was therefore difficult for these children to simultaneously recall the hidden shape's verbal name. This second technique of recognition provides a clue to understanding the complex mental reformulating that the child's mind may need to do while shifting from a visual to a more analytical way of thinking. Thinking about the shapes in terms of its properties calls for a multi-structural approach versus the uni-structural approach (Biggs & Collis, 1982) that was sufficient for the visual holistic recognition of shapes.

Shape reproduction. We examined children's abilities to copy a stimuli and to reproduce a stimuli from memory. Effect of treatment was found in the quality of reproducing the circles, measured as completing a closed curvilinear shape. No difference was found in the square reproduction task. It is interesting to note that children did not differ in their ability to identify a circle but differed in their ability to draw one.

Other activities, such as making a circle with your body, support the theory that touching one part of a shape (tactile perception) or making a shape (physical action) triggers a conceptual perception which when connected with other tactile perceptions, starts to establish relationships between perceptions, leading to a stronger representation of the shape.

Orientations. Children in the experimental group could better identify horizontal and vertical orientations in realistic pictures than the control group. Oblique orientations were confused with curved lines by both groups.

The synthesized curriculum engaged the children in making lines in different orientations with their bodies, drawing these and discussing figures that could be made from these lines. The teacher used these terms during whole group times or while discussing properties of shapes.

Patterns. While all children evinced awareness and ability to pattern, the experimental group extended their understanding of this concept and significantly outperformed the control group in completing patterns in terms of number of forms, positioning of the shapes and relative size of shapes.

The synthesized curriculum engaged the children in activities of visual decoding and mental visual coding that focused on reproducing patterns from memory as well as creating their own patterns with attribute blocks.

Summary

Visual thinking. Results from this study support the Agam program's that enhancing visual thinking skills through geometry-oriented activity may lead to more robust understanding of shape concepts. The visual thinking skills such as perceptual flexibility, visual encoding and decoding, and visual memory may have been enhanced through the synthesized curriculum, leading to a more robust understanding of the shape concepts. As done in this curriculum, and effectively so, visual thinking skills can be taught as a basic skill in order to enhance visual imagery.

Levels of geometric thinking. The synthesized curriculum enhanced children's levels of geometric thinking. At the outset, a majority of children in both treatment groups could not verbalize their rationale for shape identifications or exhibited a lower level of justification stating (for example) that a triangle is one "just because." It was an intuitive state, with children claiming they were right and not able to articulate why.

According to the van Hiele (1986) levels of geometric thinking, level 1 is visual/perceptual with children recognizing shapes by their holistic visual prototypes. At level 2, children begin to consider the attributes of shapes and are able to perceive shapes as combination of properties. A modification of these levels to include a more basic pre-representational level 0 (Clements & Battista, 1992) more closely portrays the levels of geometric thinking of children in his study. At this level, children are able to only focus on one subset of a shape's visual characteristics (such as curvilinear and rectilinear features) and not able to identify many common shapes.

At the start of this study, most children (from both groups) were at level 0, the pre-representational level. Gradually, within the synthesized curriculum, the children began to develop visual prototypes of the shapes, while almost simultaneously beginning to reason using attributes of shapes, such as 'pointy' and 'corners'. This recognition of attributes was ascribed to individual shapes without necessarily relating to all shapes that held the same attributes. Also, the children did not relate attributes to each other (such as a shape with three points must have three sides). Through repeated activities in the synthesized curriculum, children evinced growth in relating attributes and relating shapes with the same attributes. This strengthened their perceptual flexibility and led to stronger shape identification such as recognizing the square as a square, even if it was rotated. Their verbal justifications or math talk, during shape identifications provided additional evidence of their strengthening perceptual flexibility and their conceptualizations of geometric shapes.

A few of the children in the treatment group started using used attribute-based reasoning early on, in the study. But these attribute phrases (for example, 2 long sides and 2 short sides) were almost empty words that they had heard their teacher repeatedly use and hence followed her model without being able to expand on these words. As they got more immersed in the synthesized curriculum and were doing activities related to these same attributes, they sometimes stopped using the phrases and or used them more sporadically. A few of these children came back to referring to the attributes, not for all the shapes but whenever they did use them, they could expand on it and explain why they used that rationale. It is possible that the children first internalized these terms as descriptive words, unassociated to the context and the concepts and only later (after experience with the attributes) learnt to endow them with geometric meaning.

General cognitive abilities. We measured children's cognitive abilities through the Raven's Colored Progressive Matrices and the Goodenough-Harris Draw-a-Man test. The experimental group significantly outperformed the control group in the former test, suggesting that their visual thinking skills were enhanced through treatment sessions. Additionally, it afforded cognitive tools to solve problems presented in the test, resulting in an increase in performance.

Educational implications. Academic readiness and cognitive test performance require the acquisition of various prerequisite skills. In our study comparing a synthesized Agam curriculum to a more typical curriculum, kindergartners showed that they were able to acquire some of the skills that transfer to readiness and test performance. Children in the experimental group had little difficulty understanding the basic geometry concepts. They were able to use problem solving skills and visual thinking skills acquired through the intervention when engaged in activity in the classroom, in addition to completing the geometry concept test. In contrast, the control group did not show a pronounced increase in geometry concept understanding.

The Building Blocks curriculum

Building Blocks was an NSF-funded mathematics curriculum development project, designed to comprehensively address recent standards for early mathematics education for all children (Sarama & Clements, 2002). Previous articles describe research-based design model that guided its development (Clements, 2007; Sarama & Clements, 2002). Here we briefly review the basic design, then provide geometric examples.

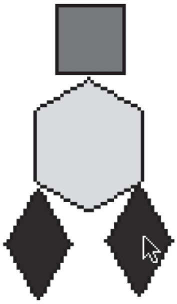
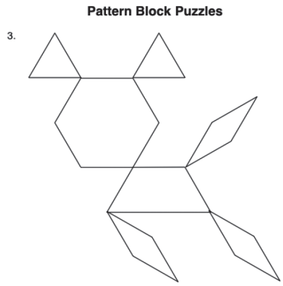
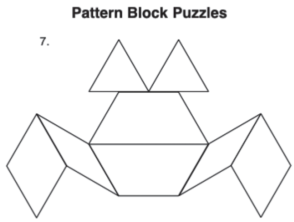
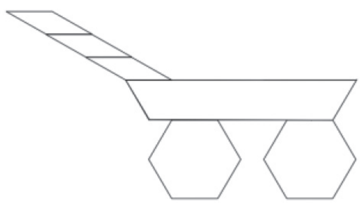
The core of Building Blocks: research-based learning trajectories

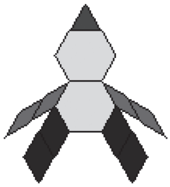
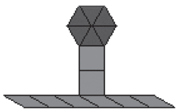
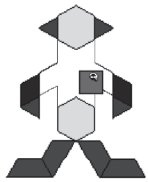
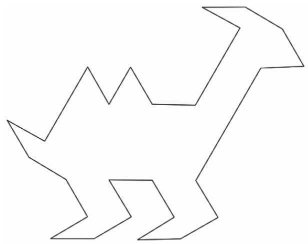
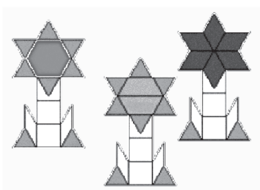
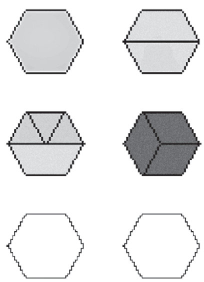
All components of the resulting Building Blocks curriculum (Clements & Sarama, 2007/2013) are based on learning trajectories for each core topic. First, each topic is examined to ensure it is appropriate and important – mathematically and generative of future learning. Secondly, empirically-based models of children's thinking and learning are synthesized to create a developmental progression of levels of thinking in the goal domain (Clements & Sarama, 2004; Clements, Sarama, & DiBiase, 2004; Gravemeijer, 1999; Simon, 1995). Third, sets of activities are designed to engender those mental processes or actions hypothesized to move children through a developmental progression.

In geometry, research such as that reviewed here supported the importance of the topic of geometric shape and spatial reasoning. It also revealed distinct levels of geometric thinking (see also Clements, 1992; van den Heuvel-Panhuizen & Buys, 2005; van Hiele, 1986). A synthesis of this corpus produced a developmental progression – the core of a learning trajectory – for young children's learning of two-dimensional geometric figures (Sarama & Clements, 2009b). Grounded in Piagetian (1967) and Van Hielian (1986) theories, this developmental progression was structured to establish more fine-grain levels than those frameworks (Clements, 1992). The progression for knowledge of geometric figures moves from increasingly sophisticated comparing (matching) through levels of recognizing and naming (including variants and difficult distractor – Figure 1), identification of the components of figures, to the understanding of properties of shapes, and finally the use of those properties when classifying and analyzing sets of geometric figures. Instruction was designed to help gain competencies at each level.

To provide an illustration, consider the related topic of shape composition. Composing of two-dimensional geometric figures was determined to be significant for children in two ways (similar for 3D shapes, which we do not discuss here). First, it is a basic geometric competence, growing from preschoolers' building with shapes to sophisticated interpretation and analysis of geometric situations in high school mathematics and above. Second, the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis (Clements, Battista, Sarama, & Swaminathan, 1997).

The developmental progression was born in observations of children's explorations (Sarama, Clements, & Vukelic, 1996) and refined through a series of clinical interviews and focused observations, then validated by comparing to previous studies (Mansfield & Scott, 1990; Razel & Eylon, 1986; Sales, 1994; Vurpillot, 1976). From a lack of competence in composing geometric shapes (the Pre-Composer level, see the first column in Figure 2), children gain abilities to use individual shapes to make a picture but with each shape playing a unique semantic role (Piece Assembler) and then to combine shapes – initially through trial and error (e.g., Picture Maker), concatenating shapes to form a component of a picture but not necessarily conceptualizing these creations as geometric shapes. At the Shape Composer level they have developed competence with angles and anticipatory imagery and thus compose shapes to intentionally create superordinate shapes (For details on other learning trajectories, see Clements & Sarama, 2014; Sarama & Clements, 2009b).

Developmental Progression	Instructional Tasks
Piece Assembler Makes pictures in which each shape represents a unique role (e.g., one shape for each body part) and shapes touch. Fills simple "Pattern Block Puzzles" using trial and error. Make a picture 	In the first "Pattern Block Puzzles" task, each shape is not only outlined, but it touches other shapes only at vertices.  Then, the puzzles moved to those that combine shapes by matching their sides, but still mainly serve separate roles.  See also software at LearningTrajectories.org.
Picture Maker Puts several shapes together to make one part of a picture (e.g., two shapes for one arm). Uses trial and error, choosing shapes using "general shape" or side length. Fills "easy" "Pattern Block Puzzles" that suggest the placement of each shape.	The "Pattern Block Puzzles" at this level start with those where several shapes are combined to make one "part," but internal lines are still available.  Later puzzles in the sequence require combining shapes to fill one or more regions, without the guidance of internal line segments.

Developmental Progression	**Instructional Tasks**
Make a picture  **Shape Composer** Composes shapes with anticipation. Chooses shapes using angles as well as side lengths. Rotation and flipping are used intentionally to select and place shapes. Make a picture 	 The "Pattern Block Puzzles" and "Piece Puzzler" activities have no internal guidelines and larger areas; therefore, children must compose shapes accurately. 
Substitution Composer Makes new shapes out of smaller shapes and uses trial and error to substitute groups of shapes for other shapes to create new shapes in different ways. Make a picture with intentional substitutions 	At this level, children solve "Pattern Block Puzzles" in which they must substitute shapes to fill an outline in different ways.  **Pattern Block Puzzles and Tangram Puzzles** Ask students how many of a certain shapes it would take to cover another shape (or configuration of shapes). Students predict, record their prediction, then try to check.

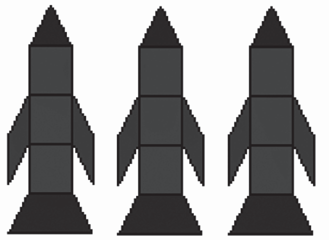
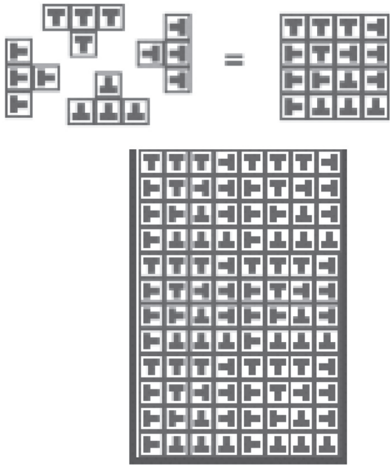
Developmental Progression	Instructional Tasks
Shape Composite Repeater Constructs and duplicates units of units (shapes made from other shapes) intentionally; understands each as being both multiple small shapes *and* one larger shape. May continue a pattern of shapes that leads to tiling. Children use a shape composition repeatedly in constructing a design or picture.	Children are asked to repeat a structure they have composed. 
Shape Composer – Units of Units Builds and *applies* units of units (shapes make from other shapes). For example, in constructing spatial patterns, extend patterning activity to create a tiling with a new unit shape—a unit of unit Shapes that they recognize and consciously construct. Builds a large structure by making a combination of pattern blocks over and over and then fitting them together.	In this “Tetrominoes” task, the child must repeatedly build and repeat superordinate units. That is, as in the illustration here, the child repeatedly built “Ts” out of four squares, used 4 Ts to build squares, and used squares to tile a rectangle. 

Figure 2. Example levels from a learning trajectory for the composition and decomposition of 2D shapes

A main instructional task requires children to solve outline puzzles with shapes off and on the computer, a motivating activity (Eylon & Rosenfeld, 1990; Razel & Eylon, 1986; Sales, 1994; Sarama, Clements, & Vukelic, 1996, note Building Blocks borrowed many other aspects and activities from the Agam program, with permission). For example, they solve geometric puzzles (see the rightmost column in Figure 2). The objects are shapes and composite shapes and the actions include creating, duplicating, positioning (with geometric motions), combining, and decomposing both individual shapes (units) and composite shapes (units of units). The characteristics of the tasks require actions on these objects corresponding to each level in the learning trajectory. For example, in Figure 2, the puzzles for Piece Assembler scaffold children's growing ability to match shapes without, and then with sides touching. Those for the Picture Maker level increasingly require that children must compose shapes and those for Shape Composer have larger areas in which angles must be attended to. Teachers are guided to use the cognitive model of each level (Sarama & Clements, 2009b) to guide formative assessment, identifying children's level of thinking and interacting with them to help them develop them subsequent level. Ample opportunity for student-led, student designed, open-ended projects are included in each set of activities.

Results from Multiple Studies – What geometry did children learn?

In an early comparison study involving only four classrooms, effects on geometry were large (effect sizes, > 1 SD, Clements & Sarama, 2007). Examining specific topics, the relative effects on turn and congruence were small, but effects on construction of shapes and spatial orientation were large. The largest relative gains in geometry were achieved on shape identification and composition of shapes (gains in composition were four times as large as those of the control group, Clements & Sarama, 2007). A qualitative study of children's learning in the Building Blocks classrooms showed that children felt quite powerful knowing and applying definitions of triangles (Spitler, Sarama, & Clements, 2003). One preschooler said, "That's not a triangle! It's too skinny!". However, his Building Blocks friend responded, "I'm telling you, it is a triangle. It's got three straight sides, see? One, two, three! It doesn't matter that I made it skinny".

A second study was a cluster randomized-trials evaluation involving 36 preschool classrooms randomly assigned to one of three conditions: Building Blocks, comparison (a different preschool mathematics curriculum), or control (Clements & Sarama, 2008). The Building Blocks group scored significantly higher than the comparison group (effect size, .47) and the control group (effect size, 1.07). On geometry, both intervention groups scored higher than the control group, with little difference between them, on identifying shapes and constructing shapes, although the Building Blocks group increased in the frequency of completely correct constructions more than the other two groups (again, these comparisons involve descriptive statistics only, not inferential tests). The Building Blocks group scored higher than both the comparison and control groups on comparing shape and on shape composition, mostly due to the Building Blocks group generating fully correct solutions. They also increased substantially more than the other groups in using more sophisticated strategies, such as rotating shapes into the correct orientation before placing them on the puzzle, searching for specific shapes with intentionality, and, in general, solving the puzzle systematically, immediately, and confidently.

Finally, we also evaluated Building Blocks' effectiveness in a larger cluster randomized trial design involving 42 schools serving 1375 preschoolers in 106 classrooms. Teachers implemented the intervention with adequate fidelity. The overall positive effect was moderate to large (.72 SD); we did not analyze its effects on geometry separately. For this paper, we calculated effects for the geometry sections of the assessment. HLM analyses revealed a significant difference for the Experimental versus Control conditions (beta=.273, $p<.05$, effect size, .61). Children within the Building Blocks condition demonstrated higher ability scores on geometry than children in the control condition.

General discussion. Supporting previous studies, emphasis on the learning trajectory for geometric topics in the Building Blocks curriculum led to effects, some equivalent to benefits often found for individual tutoring. In the second study, the Building Blocks curriculum made the most substantial gains compared to both a non-treatment and another preschool math curriculum in shape composition and several other geometric topics. Especially because the comparison curriculum also included all these geometric topics, such as shape composition activities, we believe that the greater gains caused by the Building Blocks curriculum can be attributed to its explicit use of the sequenced activities developed from, and the teachers' knowledge of, learning trajectories. These learning trajectories explicate the mathematical concepts, principles, and processes involved in each level of the trajectory and the relationships across levels, including the components of geometric shapes (e.g., correct definition of "side") as well as relationships between components (e.g., sides forming a right angle) and shape classes (e.g., a square as a subcategory of rectangle and justification for this based on properties). The learning trajectories are also designed to develop teachers' knowledge of students' developmental progressions in learning that content (moving from intuitively recognizing shapes as unanalyzed visual wholes, to recognizing components of shapes, to hierarchically classifying shape categories) and to inform teachers of the rationale for the instructional design of each (e.g., why certain length sticks are provided to children with the challenge to build specific shapes).

Research on math talk during play

In recent years, studies on math learning and teaching have identified classroom discourse, particularly, teachers' math talk as a significant contributor to children's mathematical thinking and development. The frequency and types of math talk exuded by teachers has been associated with math learning in early childhood classrooms (e.g., Klibanoff, Levine, Huttenlocher, Vasilyeva, & Hedges, 2006). These results support Vygotsky's (1934/1986) theory that language nurtures and propels learning by clarifying their developing understanding, supporting memory, focusing attention and ultimately refining problem-solving. Such math talk during free play has been observed and documented in research (e.g., Cooke & Buchholz, 2005). Other research has looked at specific play scenarios such as building with blocks (Hanline, Milton, & Phelps, 2001; Trawick-Smith, Oski, DePaolis, Krause, & Zebrowski, 2016) playing board games (Ramani & Siegler, 2008), and making puzzles (Levine, Ratliff, Huttenlocher, & Cannon, 2012). We conducted a series of math talk studies during play to further isolate specific types of Math-talk that was particularly effective in enhancing children's math understanding and growth. We summarize key findings from these studies below as they relate to geometry.

Naturally occurring math talk in preschool play

Preschool teachers' discourses during play centers were recorded (a total of 32 hours) and transcribed to determine their math talk patterns during naturally occurring play interactions, i.e., within child-initiated activities, as opposed to teacher-set curricula. Results indicated that geometry was one of three math content (others being number and measurement) that teachers naturally included in their discourses. Additional findings added deeper educational implications in that not all of these discourses were a good fit to support children's play behaviors. For example, teachers' efforts to generate discourse rich in geometry was sometimes strained and did not naturally fit within the child's play scenarios and at times, continued for far longer durations than of interest to the child.

Relationship of math talk to math learning

Following up on the previous study, we further examined associations between these math talk categories and growth in math learning during a full year of preschool. Forty-seven preschoolers' math knowledge was assessed at the beginning and end of their school year. Each child was recorded during five, 20-minute play periods and edited to tease out all the teacher-child interactions pertinent to math talk. Results yielded several significant findings. One, good-fit interactions (math talk that matched the child's need in terms of content, context and levels of guidance) significantly impacted their growth. Two, teachers engaged in more number talk than geometry and the former was significantly related to children's growth. That geometry related math talk was not related to math growth needs to be understood within the context of the infrequency of geometry talk. Teachers naturally engaged in number talk much more than geometry talk. Classroom variations were detected that further illumine these results. Teachers who used more open-ended questions, posed more problem-solving questions, and also used more geometry-related math talk had greater impact on the children's posttest math scores.

Effects of math talk training in urban child care classrooms

Following up on the previous two studies, particularly the finding that geometry math talk was less frequent, we conducted the next study to determine the effects of training on teachers' math talk. Thirteen preschool teachers and their 66 three- and four-years old preschoolers were involved in this study. Teachers participated in an eight-hour training program to increase math talk followed by documentation of their math talk by content areas. Initial data indicated a heavy use of number talk but after seeing the data on their own math talk distribution by content areas, teachers intentionally increased their geometry and measurement related talk. Concomitant with that, children's math assessment, especially the two measures on spatial knowledge and shape recognition, showed significant growth.

Math talk by categories and purpose

In the next study, we drilled down teachers' math talk to determine the purposes and to identify key sub-categories within the math content areas. Examining over 100 math talk clips, several purposes were identified including instructing, guiding and posing open/closed problems. Examining the sub-categories, it was determined that geometry math talk included spatial knowledge, shape identification, shape composition, attributes of shape and transformations, but not with equal frequencies.

Impact of math talk sub-categories and purpose on math learning

In the fifth study, we examined the impact of these sub-categories of math talk, purposes and syntax (statements vs. open and closed questions) on children's math learning. Forty preschoolers' math knowledge was assessed using Research-based Early Mathematics (REMA, Clements, Sarama, & Wolfe, 2008/2016) at the beginning and end of their preschool to determine impact of the teachers' math talk on their math growth. Overall math talk did impact math growth. Closely examining the sub-categories of geometry math talk, only discourse that focused on the attributes of shapes had a significant impact on children's math learning. Math talk on shape identification did not have a significant impact.

Overall discussion

Findings of these five studies carry several implications for geometry learning and teaching. First, teachers' math talk during play is important for nurturing children's growth but needs to be closely connected and in sync with what children are currently doing and thinking. Second, geometry math talk is not as frequently seen in classrooms. Teachers and professional development practices need to explicitly include an emphasis on geometry math talk. This supports Vygotsky's view that language supports thinking. It is also consistent with the work of Weisberg, Hirsh-Pasek, and Golinkoff (2013) that conversations that are congruent with and support, rather than interrupt, children's play and thinking are most powerful. Third, geometry math talk appears to be more closely intertwined with open-ended constructive classroom cultures and less visible in classrooms that focus on basic skills. This confirms findings of previous studies that kindergarten teachers equate math learning in early childhood as simple counting and rarely challenge students to think deeply about geometry (Engel, Claessens, Watts, & Farkas, 2016). Fourth and perhaps most critically, geometry talk that focuses on shape identification (most frequently seen in classrooms) is not as effective as discourse that drills down into the properties of shapes.

Conclusions and implications

The studies discussed here show that young children are capable of learning geometric ideas and processes to a surprising degree and depth. We find that children can learn richer concepts about shape if their educational environment includes six features: varied examples and nonexamples, a wider variety of shape classes, and a broad array of geometric tasks, movement from "Sensory-Concrete" to "Integrated-Concrete" cognition (Clements, 1999; Clements & McMillen, 1996; Sarama & Clements, 2009a, 2016; Sarama, Clements, & Vukelic, 1996), rich math talk about shapes, their attributes and properties, and the teacher's use of learning trajectories. We discuss each of these in turn.

First, educators should ensure that children experience many different examples of a type of shape, so that they do not form narrow ideas about any class of shapes. Use of prototypes may bootstrap initial learning, but examples should become more diverse as soon as possible. Showing nonexamples and comparing them to similar examples help focus children's attention on the critical attributes of shapes and prompts discussion. This is especially important for classes that have more diverse examples, such as triangles.

Second, curricula and educational activities should include a wide variety of shape *classes*. Early childhood curricula traditionally introduce shapes in four basic level categories: circle, square, triangle, and rectangle. The idea that a square is not a rectangle is rooted by age five. We suggest presenting many examples of squares and rectangles, varying orientation, size, and so forth, *including squares as examples of rectangles.* If children say, "that's a square" teachers might respond that it is a square, which is a special type of rectangle, and they might try double naming ("it's a square-rectangle"). Older children can discuss "general" categories, such as quadrilaterals and triangles, counting the sides of various figures to choose their category. Also, teachers might encourage them to describe why a figure belongs or does not belong to a shape category. Then, teachers can say that because a triangle has all equal sides, it is a special type of triangle, called an equilateral triangle. Children might also "test" right angles on rectangles with a "right angle checker". Computer environments can also engage and develop children's thinking about relationships between classes of shapes, including squares and rectangles. In one large study (Clements, Battista, & Sarama, 2001), some kindergartners formed their own concept (e.g., "it's a square rectangle") in response to their work with Logo microworlds. Further, children should experiment with and describe a wider variety of shapes, including but not limited to semi-circles, quadrilaterals, trapezoids, rhombi, and hexagons.

Third, educators should challenge children with a broad array of interesting tasks. Experience with manipulatives and computer environments are often supported by research, if the experiences are consistent with the implications just drawn. Activities that promote reflection and discussion might include building models of shapes from components. Matching, identifying, exploring, and even making shapes with computers is particularly motivating (Clements, 2003).

Fourth, children should be guided to move from Sensory-Concrete to Integrated-Concrete cognition (Clements, 1999; Clements & McMillen, 1996; Sarama &

Clements, 2009a, 2016; Sarama, Clements, & Vukelic, 1996). The studies cited here, such as those from the Agam program and researched-based learning trajectories, support development from Sensory-Concrete implicit levels at which perceptual supports are *necessary* and fundamental to later learning (i.e., building strong concept images, Vinner & Hershkowitz, 1980), and reasoning may be restricted to limited cases, to more explicit, verbally-enhanced generalizations and abstractions that characterize Integrated-Concrete understandings (involving *internalized* mental imagery and linked verbal schemes that generate operations and abstractions that are increasingly sophisticated and powerful).

Fifth, educators should encourage children's descriptions while encouraging the development of language through math talk. Visual (prototype-based) descriptions should, of course, be expected and accepted, but attribute and property responses should also be encouraged. They may initially appear spontaneously for shapes with stronger and fewer prototypes (e.g., circle, square). Again, they should be especially encouraged for shape categories such as triangles. Children can learn to explain why a shape belongs to a certain category – "It has three straight sides" or does not belong ("The sides aren't straight!"). Eventually, they can internalize such arguments; for example, saying, "It is a weird, long, triangle, but it has three straight sides!".

Sixth, teachers should understand and teach with learning trajectories, which connect mathematical goals, children's thinking and learning levels, and instruction. They support formative assessment, helping teachers identify children's level of thinking and interact with them to maximize their learning.

In these ways, children move through developmental progressions, building intuitions and perceptually-based competencies and then explicating these early developments through math talk to connect verbal and abstract thinking – building integrated concrete knowledge of geometry and spatial reasoning. Throughout the early childhood years, it seems not only appropriate, but essential, to introduce, expand, and guide children through this learning process.

Notes

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