Fostering process-object transitions and a deeper understanding in the domain of number

Promovendo transições processo-objeto e uma compreensão mais profunda no domínio dos números e operações

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Abstract. The gist of this article is that a shift is needed towards a mathematics curriculum in which teaching for understanding is the main objective. Before elaborating on what such a shift might entail, two compelling arguments for making such a change are presented. One is based on research, which showed that a one-sided emphasis on skills induced the teaching of isolated, topic-specific, skills – leading to a low level of proficiency. The other argument is grounded in the observation that the role of mathematics in society is changing and that, as a consequence thereof, the importance of mastering routine skills diminishes, while the need for mathematical understanding grows. On the basis of these two arguments, a shift is advocated, from an emphasis on skills, to an emphasis on understanding. This is connected to the thesis that deep mathematical understanding can only be achieved when students construct mathematical objects by reifying mathematical processes. For the domain of number, the idea of mathematical objects is linked to the notion of junctions in networks of number relations. The core of the article is an exploration of what mathematics education in the number domain might look like if it would be organized along the line of process-object transitions, and how number relations can be used for solving the kind of number tasks that are commonly solved with standard procedures. This exploration is closed with a sketch of a potential instructional sequence for addition and subtraction up to 100.
Keywords: process-object transition; addition; subtraction; conceptual understanding; skills.

Resumo. A essência deste artigo é que é necessária uma mudança para um currículo de matemática em que o objetivo principal é o ensino para a compreensão. Antes de discutir o que essa mudança pode acarretar, são apresentados dois argumentos convincentes para fundamentar essa mudança. Um é baseado na investigação, que mostra que a ênfase unilateral nas técnicas induz o ensino de técnicas isoladas, específicas de um tópico, levando a um baixo nível de proficiência. O outro

argumento fundamenta-se na observação de que o papel da Matemática na sociedade está a mudar e que, como consequência disso, a importância de dominar as técnicas rotineiras diminui, enquanto a necessidade de compreender a Matemática aumenta. Com base nestes dois argumentos, defende-se uma mudança, de uma ênfase nas técnicas rotineiras para uma ênfase na compreensão. Isto liga-se à tese de que uma compreensão matemática profunda só pode ser alcançada quando os alunos constroem objetos matemáticos reificando processos matemáticos. Para o domínio dos números e operações, a ideia de objetos matemáticos está ligada à noção de articulações em redes de relações numéricas. A parte central do artigo é uma exploração de como seria a educação matemática no domínio dos números e operações se fosse organizada ao longo da linha de transições de processo-objeto e de como as relações numéricas podem ser usadas para resolver o tipo de tarefas numéricas comummente resolvido com procedimentos padrão. Esta exploração termina com o esboço de uma potencial sequência de ensino para a adição e a subtração até 100.
Palavras-chave: transição processo-objeto; adição; subtração; compreensão conceptual; técnicas rotineiras.

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Introduction

Empirical research in the Netherlands shows that teaching isolated skills falls short of expectations in terms of mathematical proficiency. Learning sets of individual solution methods, which are tailored to specific cases, undermines flexibility, and it results in a multitude of procedures, which are mixed up by the students. Furthermore, teaching isolated skills may go at the expense of teaching for understanding.

Taking a broader perspective, we may question the need for routine skills in the 21st century. Nowadays, mathematical procedures taught in primary, secondary and tertiary education can be done by computers. This has a huge impact on what mathematics students will need in later life. As computers are taking over the bulk of mathematical work, the need to understand the relevant mathematics will be growing simultaneously. Furthermore, the extent of mathematical skills that students need to have at their immediate disposal will decrease. As a consequence thereof, instruction time comes available for teaching for understanding.

Against this background the question arises, what "understanding" we should aim for. In this article, we search for an answer to this question in the domain of number. We follow a host of renown scholars, who claim that transitions of processes into objects, which are again used in new processes, not only reflect the way mathematics emerged in history, but also lays at the heart of how mathematicians think. We argue that mathematics education should be organized as a sequence of successive process-object transitions. When zooming in on the domain of number, mathematical objects can be seen as junctions in networks of number relations. Flexible use of number relations in turn, may be used as a basis for solving a wide variety of tasks in the domain of number. And, thus, offer a potential alternative for standardized procedures.

This brings us to the core of our article: the exploration of how process-object transitions might be worked out in the domain of number. In doing so, we aim at offering an approach that generates a deeper understanding and can serve as an alternative for the common approach of teaching procedures.

The article is organized as follows. First, we describe the research in the Netherlands that showed the limitations of teaching isolated skills. Next, we discuss the implications of computerization. This leads us to the conclusion that a shift towards understanding is in order. We subsequently discuss what form of understanding one should aim for in future mathematics education. In connection with this, we discuss the central idea of process-object transitions. Next, we elaborate on the idea of using networks of number relations for solving number tasks, which are commonly solved with standard procedures. The idea of solving number problems by using number relations will be illustrated with addition and subtraction up to 20, and multiplying fractions. To give an idea of what instruction, aiming at process-object transitions, might look like, we sketch an instructional sequence for addition and subtraction up to 100, which we base on two design-research projects and evaluative research data. We close by reflecting on the potential of the proposed change for a curriculum tailored to the future and the feasibility of designing corresponding curricula.

The limitations of thinking in isolated skills

A series of national surveys of proficiency in mathematics at the end of primary school in the Netherlands showed a mixed image. In some domains, there was improvement, in others there was a decline (Janssen, van der Schoot, & Hemker, 2005). The fact, however, that there was a decline in procedural skills was picked up by the media and formed the start of fierce debates on the quality of mathematics education and its results. This was reason for three researchers to investigate specific skills, respectively, subtraction up to 100, operations with fractions and algebraic skills (see, Gravemeijer, Bruin-Muurling, Kraemer, & van Stiphout, 2016).

Kraemer (2011) studied subtraction up to 100 in grade 3. Building on the literature, he discerned three strategies, which were investigated in questioning 300 Grade 3 students orally. It showed that,

- the string method, which involved counting on or counting down in chunks (for instance, solving 65 – 38 = …, via 65 – 30 = 35, 35 – 5 = 30, 30 – 3 = 27), was used by most of the students (57% – 74%)[1], was effective (82% – 91% correct), and was used flexibly;
- the method of splitting tens and ones (e.g., 68 – 45 = …; 60 – 40 = 20, 8 – 5 = 3, answer 20 + 3 = 23), generated many incorrect answers (65% – 42% correct);

- the reasoning method, using arithmetical properties, also generated many incorrect answers (50% – 31% correct).

Kraemer (2011) concluded that the students had a limited understanding of tens and ones, and little understanding of the relation between addition and subtraction. Subsequently, he investigated the Dutch textbooks. He showed that the textbooks focused on the string method and gave only very limited attention to splitting and reasoning. He conjectured that this choice was probably motivated by the fact that the string method worked so well. The students, however, spontaneously started using splitting and reasoning methods, which they did not master.

Bruin-Muurling (2010) investigated operations with fractions in primary and secondary school. She showed that students were struggling with fractions well into secondary school. This, for instance, was shown in the fraction skills of 9th grade students in pre-university and pre-vocational secondary schools (N=347); tasks involving

- addition and subtraction with non-related denominators were only mastered by the best students;
- multiplications, such as "$\frac{4}{7}$ of 35 euro" or "$5 \times \frac{41}{5}$", were mastered by half of the students;
- division in contexts were mastered by half of the students;
- bare division tasks were mastered by a quarter of the students.

Analyzing the data, she concluded that students lacked understanding of the underlying concepts, such as *unit*, *fraction as a number*, and *fraction as a division*. In addition, the students did not grasp the relation between fractions and the operations multiplication and division.

Bruin-Muurling (2010) also investigated how multiplying fractions was addressed in primary and secondary textbooks. The primary-school textbooks start with developing meaningful calculation methods, grounded in phenomena that are familiar to students. However, the informal solution procedures emerging from solving contextual problems formed the starting point for teaching number-specific solution procedures. In the context of finding the total volume of 16 cartons of $\frac{3}{4}$ liter milk, for instance, $16 \times \frac{3}{4}$ might be solved by adding, $\frac{3}{4} + \frac{3}{4} + \cdots + \frac{3}{4}$, 16 times. This was turned into to the procedure of solving tasks such as $16 \times \frac{3}{4}$ by repeated addition. When calculating $\frac{3}{4}$ of 16 kilograms, it is more obvious to first take one quarter and then take 3 of these quarters. This was turned into $\frac{3}{4} \times 16$ is solved via $16 \div 4 = 4$; $3 \times 4 = 12$. These embeddings in informal knowledge led to two separate solution procedures: one for "a whole number times a fraction", and one for "a fraction times a whole number". Apart from these two, the textbook developed other procedures for the multiplication of two proper fractions and for multiplying a whole number times a mixed number. However, the next step was lacking. The textbooks did not address questions such as "Why is it that $16 \times \frac{3}{4}$ equals ¾ of 16?" or "Why does $\frac{3}{4} + \frac{3}{4} + \ldots \frac{3}{4}$ get you the same answer as (16 ÷ 4) x 3?"

The secondary textbooks quickly introduced the formal rule "multiply the nominators and multiply the denominators", which was "proven" with one example with an area model. Understanding this rule, however, presupposes an understanding of fractions as mathematical objects/rational numbers. Whereas the students most likely still conceived fractions as part-wholes, tied to some measurement unit – whether meters, liters or pizzas, the new rule was just another procedure for the students. By then, they had learned five rules, which they started to confuse with each other. In short, in the researched textbooks, fraction multiplication was compartmentalized, as Engeström (1991) calls this phenomenon.

In algebra, Van Stiphout (2011) found a similar pattern. The students developed a limited set of solution procedures, and lacked understanding and flexibility. This was in line with the textbooks, which focused on procedures for isolated tasks – albeit after a conceptual start.

Generalizing over the tree studies, Gravemeijer et al. (2016) coined the term *task propensity*, which they define as "The inclination of both teachers and textbook authors to think of instruction in terms of individual tasks that have to be mastered by the students" (p. 35). This propensity seems to seep into mathematics instruction, because of the attraction of methods that produce correct answers, the need to practice, and the common approach of assessing proficiency on the basis of individual tasks. Resnick and Hall (1998) make a similar observation, which they relate to the popular view on learning that is very similar to classic associative theories of learning from the time of Thorndike. According to such theories, knowledge consists of connections between mental entities, and learning is a matter of creating and strengthening these bonds. This leads to the pedagogy of frequently testing individual items to see if the corresponding bonds are formed, and subsequently training the bonds that are not yet formed. A serious consequence of task propensity is that students become focused on task characteristics, do not see the bigger picture, and do not generalize.

In summary, we may conclude that research shows that teaching isolated skills – as appeared to be common in the Netherlands – does not work. Nevertheless, this was in contrast with what Dutch schools, in general, aimed for, especially the ones following a Realistic Mathematics Education (RME) approach. And, as far it concerned the conceptual foundation, Dutch textbooks were in line with this intention. The textbooks started in an RME fashion, but they finished with presenting tasks in a compartmentalized fashion. They shifted towards practicing compartmentalized individual skills – probably under influence of tasks propensity. We may add that – at that time – RME aimed at standardized procedures as potential endpoints, as basic skills formed the core of the mandated curriculum. Which they still do. We may question, however, if this still fits with the demands of today's society.

Mathematics education for the future

Our society is changing fast under the influence of computerization (Brynjolfsson & McAfee, 2014; Levy & Murnane, 2005). Workers, and citizens in general, have to be able to both work with computerized apparatus, and deal with computer-generated information. Mathematics has a central place in this development, as (1) more sophisticated mathematics is getting used, and (2) mathematics is increasingly carried out by machines (Wolfram, 2014). This self-evidently has a series of consequences for mathematics education, which we will elaborate in the following.

In relation to the societal changes as an effect of globalization and computerization, many point to the, so-called, 21st century skills (Voogt & Pareja, 2010). These concern competencies such as flexibility, creativity, critical thinking, taking different perspectives, and considering multiple solutions. These competencies may well be addressed in mathematics education. In fact, they have been advocated by the mathematics education community for decades – especially in connection with developing mathematical insights (National Council of Teachers of Mathematics, 1989). In order to develop a deep understanding, students would have to solve problems, construct and discuss ideas, and look critically at those ideas. In addition to the aforementioned general skills, the pervasive role of computerization also induces changes in what mathematical topics are important in our society. Statistics is maybe the most cited example, as the growing computing power allows for increasing possibilities to assemble and process data (Brynjolfsson & McAfee, 2014). But we may also mention 3D geometry, measuring (physical, societal, economical, and other phenomena), variables and functions, calculus, cryptography and algorithms. The growing computational power of computers further has a consequence that more complex problems can be tackled. This results in higher demands on modeling competencies (OECD, 2016), which involve modeling problems, working with ready-made models, understanding the relation between model and reality, and also identifying (new) opportunities for applying mathematical models, and being aware of the limitations of a mathematical approach. Also, for an average citizen, it will be important to have a sense of the impact and significance of models.

The pervasive role of computerization further has as a consequence that, increasingly, basic calculations are delegated to apparatus. We may argue that people using those devices should – on a global level – understand the mathematical operations that the computer is carrying out (Hoyles, Noss, Kent, & Bakker, 2010), among others, in order to interpret the output correctly, and to grasp how the input influences the output. But also, to be able to communicate with colleagues and customers about the algorithms and models that are implemented in the devices that are used. In addition, one has to be aware of the impact of the use of algorithms (O'Neil, 2017).

Another consequence of computers doing all basic calculations is that the competency of carrying out standard mathematical procedures by oneself diminishes in significance (Wolfram, 2010), although understanding stays relevant. Moreover, a certain level of procedural proficiency will still be needed, among others, in order to be able to develop conceptual mathematical understanding. We may, however, presume that speed in executing procedures will be less an issue, as machines will do most calculations. The latter also implies that the complexity of tasks for which standard procedures have to be mastered may be limited. Similarly, the range of numbers for which students will have to develop fluency in arithmetic can probably be diminished. In general, one will use electronic tools to derive precise answers for more extensive calculations. What remains is the understanding of such standard procedures.

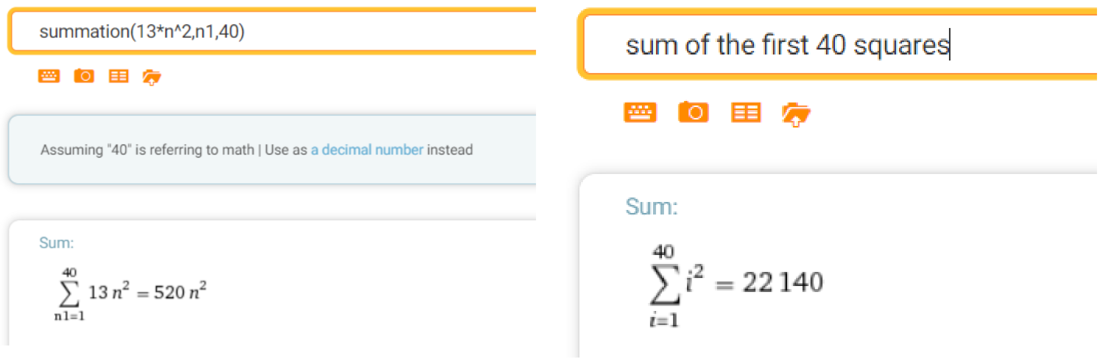
Furthermore, computational skills will still be needed to check the outcomes of the mathematical processes computers carry out (Gravemeijer, Stephan, Lin, Julie, & Ohtani, 2017). Computers are free from computational errors, but mistakes in input or chosen operations are easily made. So, it becomes important to be able to judge whether the output is reasonable. This requires skills that are different from those needed for routinely executing standard procedures (see Figure 1 for some examples).

Global calculations are not only relevant for checking answers. They are also useful beforehand, for instance when the exact numbers are not known yet, or potential consequences of certain actions have to be asserted as approximations. The quality of the approximation that is required, how rough or close it has to be, depends on what is needed in the context. In practice, weaker students might limit themselves to very rough approximations, if that would be an approximation they trust.

Mathematical understanding

Reflecting on the issues we discussed up to now, we may conclude that there are various arguments to shift the emphasis in the curriculum from skills to understanding. Firstly, a one-track emphasis on skills may result in a focus on isolated skills, and research in the Netherlands showed that teaching isolated skills has a negative effect on the development of both skills and understanding. Secondly, mathematical understanding gains in significance in the digital society. Here, we want to add that understanding not only becomes more important, but the computerization of mathematical operations also creates more room for fostering understanding in mathematics education. Historically, there has always been a tension between the two goals of mathematics education: (a) mastery of procedural skills, and (b) understanding the underlying mathematics. In practice, the need for mastery of procedural skills mostly won out.

Sometimes it is sufficient to look at the magnitude, which in simple cases may be determined by looking at the number of digits. If a narrower range is needed, global calculations may be the most suitable approach. For arithmetic this requires flexibility with number relations. We may illustrate this with a (very) simple example. Suppose the device has to calculate 16 × 127. Then one might think of 8 × 125 = 1000 and conclude that 16 × 127 is more than 2000. Another option might be to use 15 × 120 = 30 x 60 or 1,5 × 1200, which offers a broader estimate. The basis for this kind of calculations is formed by networks of number relations (such as 8×125=100, 2×15=30, 120=2×60), arithmetical properties and operating with powers of 10. Those number relations, of course also encompass relations between fractions, decimals, percentages and proportion (as in ¾ = 0,75 ⇔ 75% and 3 out of 4).



The two examples are inputs in Wolfram Alpha intended to calculate the sum: 13 × 1^2 + 13 × 2^2 +….. +13 × 40^2. In the first example a small syntax error was made, resulting in a different answer than expected. The n^2 in the answer may be the first sign that something is wrong. Recognizing that 520 = 40 × 13 helps to see that 520 n^2 = 40 × 13n^2, and thus that the software interpreted the input as adding 40 times 13n^2 together.

Figure 1. Global calculations

Thus, in the 21st century, this balance can – and should be – shifted in favor of mathematical understanding. In addition, it seems very likely that computers can also help to foster understanding (Gravemeijer, 2010; Kaput, 1995; Wolfram, 2010). However, *mathematical understanding* may mean different things for different people. We therefore have to clarify what kind of mathematical understanding we want to advocate for. We will do this by contrasting two types of mathematical understanding one might aim for.

Type 1. Learning procedures with understanding

We may characterize the first type of understanding one might aim for as student understanding of procedures. Instruction directed at understanding procedures is – similar to the most-bare form of teaching procedural skills – directed at performing the procedure. The difference lies in a focus on understanding of both the various parts of the procedure, and how these parts are connected. Nevertheless, mastering the procedure is still the end-goal. Understanding is seen as supportive for reaching this mastery and for retention. In addition, the goal is also to foster flexibility and transfer. Type 1 understanding comprises grasping the rationale for the operation or standard algorithm.

As an example, we may consider the column algorithm for addition (and subtraction) – while limiting ourselves in this description to 2-digit addition and subtraction for the sake of convenience. In the course of the years, several approaches have been designed to foster understanding of this algorithm. A common approach is to offer a notation scheme in which the *tens* and *ones* columns are explicitly separated, often introduced with manipulatives such as Dienes blocks, with wooden bars representing tens and small blocks representing individual units. Here, carrying corresponds to exchanging 10 small blocks from the ones-column, for a ten-bar in the tens-column, and thus incrementing the number of *tens* in that column.

According to Resnick and Omanson (1987), the link between the blocks and the notation scheme can be reinforced in the so-called *mapping instruction* (which requires the students to do the problems both with the blocks and in writing, while maintaining a step-by-step correspondence between the blocks and the written symbols). We might, however, argue that this approach neglects the deeper understanding, which is needed to make a conceptual connection between the manipulative materials and working on a symbolic level (Gravemeijer, 1991). This is why we advocate for a different type of understanding.

Type 2. Developing a deep mathematical understanding

This type 2 understanding is directed at the mathematical core. It concerns understanding the concept of the operation, the underlying structures, and the way that the concept fits in mathematics as a whole. This includes the creative and ambiguous side of mathematics, not merely the procedural side.

In the case of the column algorithm, type 2 understanding is strongly related to the conception of numbers as mental objects. The latter concern the decimal units (such as hundreds, tens, and ones). For addition and subtraction of 2-digit numbers, for instance, students have to construct *tens* as mathematical objects. This means that, where young students are initially only able to see *ten* as either one ten, or ten ones, they eventually have to come to see *ten* as both one ten and ten ones at the same time. Thus, *ten* has to become a mathematical object for them. This process of creating a new (numerical) unit consisting of a set of smaller units, is denoted *unitizing* (Fosnot & Dolk, 2001). A new unit is created out of a set of smaller units, and becomes an object in and of itself, which the students can act and reason with[2].

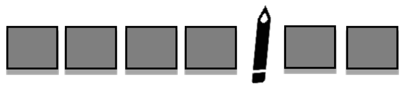
This means that numbers up to 100 are seen as mathematical objects that can be decomposed and (re)composed in a variety of sets of tens and ones, e.g., 45 = 4 tens and 5 ones, or 3 tens and 15 ones, etc. Understanding column algorithms for bigger numbers, of course, also requires the construction of *hundreds*, *thousands*, and so forth as units. Unitizing is the crucial construct that constitutes the deeper understanding, which is lacking in the type 1 understanding of the column algorithm. As a consequence, students have no way of linking manipulatives and symbols conceptually. As an additional remark, we may note that type 2 understanding surpasses the level of splitting in tens and ones. It is much broader, in the sense that numbers are eventually thought of as mathematical objects that can be decomposed and (re)composed, in arbitrary ways. Decomposing 45, for example, is not limited to 45 = 4 tens and 5 ones, or 3 tens and 15 ones, and suchlike; 45 may also be split into 22 + 23, or 9 + 9 + 9 + 9 + 9, etc. Moreover, type 2 understanding of adding and subtracting 2-digit numbers signals a broader understanding of algorithms and the conceptualization of numbers as objects. Understanding how the concepts and structures underlying the column algorithm fit mathematics as a whole becomes especially clear, when expanding the procedure to numbers larger than 2-digits.

It might seem off topic, but we also want to mention the beauty of the procedure of column arithmetic itself. The power of such algorithms or procedures is so typical for mathematics that we think that it should get more attention. The column algorithm, for example, works for all integers, regardless their length. In its core, it capitalizes on the structure of the decimal number system in combination with the analogies in the addition and subtraction regardless the unit that is used. With the latter we mean the analogy between 2+3, 20+30, 200+300. This principle also holds for 2 sevens + 3 sevens = 5 sevens or any other unit, but is especially relevant for powers of 10 in the case of column algorithms. When advocating for understanding of mathematics, we aim for this second type of understanding, which requires a corresponding form of instruction.

Understanding and what it means to do mathematics

Underlying the difference between type 1 and 2 understanding is one's view on what mathematics is, or, in other words, what it means to do mathematics. Where type 1 understanding fits with a view of mathematics as performing mathematical techniques, type 2 understanding represents a view that also comprises other aspects of doing mathematics, such as striving for generality, exactness, brevity, and certainty, which we will elaborate in the following.

These characteristics already play a role in developing understanding in early arithmetic, and are not only reserved for advanced mathematics courses. In early arithmetic, for example, students have to come to see the connection between addition and resultative counting, as Grey and Tall (1994) point out. They observed that it is essential that students see counting on as a curtailment of first counting two sets individually, and then counting the total. In this manner, counting on comes to the fore as a means of proving that a certain numerical relationship holds. This proving often stays rather implicit. We argue that making proving a more explicit topic of conversation is essential in developing a mathematical attitude[3] and type 2 understanding. Opportunities can be found everywhere, also when students are still doing mathematics at an informal level, for instance, when exploring all possible ways of splitting a (small) number, and trying to prove you have found all solutions that are possible. This can lead to a *systematic search* for all possibilities and the understanding that, by doing things systematically, you can get certainty that you did not miss an option (see also Cobb, Boufi, McClain, & Whitenack, 1997; Van der Brink, 1989). A systematic search also raises critical questions: “Is 7-0 a real split of 7?”. Sometimes this option is needed to complete the list and in other situations it is not. And “are 3-4 and 4-3 the same?”. This, of course, also depends on the situation.



4 + 2 = 6 4 = 6 - 2

6 - 4 = 2 2 + 4 = 6

Figure 2. Different perspectives on this simple drawing lead to different sums to describe the drawing

As a follow up on the activity of splitting in a simple setting, Figure 2 could be taken as a starting point for a mathematical discussion about the relation between 4+2=6, 6-4=2, 6-2=4, and 2+4=6. Just by looking at the situation from different perspectives, students may discover that addition and subtraction act as each other’s opposites. Students will encounter other such relations between multiplication and division or square and square root, and may come to see the connection between these relations and develop an understanding of inverse operations. Addressing this “going back and forth” as a general principle may foster deeper understanding, later in the curriculum, for instance, of how integration and differentiation also constitute inverse operations. This example can be seen as one of many general ideas, which relate to a variety of mathematical topics and permeate mathematics, but are not always sufficiently addressed.

With the above exposition, we want to show how a given view on doing mathematics and type 2 of understanding are interwoven. In this view on “what is means to do mathematics”, we find the core of mathematics and thus the goals we aim for when we think about type 2 understanding. The other way around, reaching type 2 understanding results from doing mathematics. We want to highlight the idea of looking at the situation from different perspectives. This leads to insights in how different perspectives relate. At the same time, being able to see things from different perspectives may be considered essential to mathematics (Antonsen, 2015).

Returning to the column algorithm, we may note that constructing algorithms can be seen as a typical mathematical activity. In *algorithmetizing*, we find all four characteristics (striving for generality, exactness, brevity, and certainty) that we mentioned at the beginning of this section. Algorithms, procedures, and rules make solving mathematical problems and organizing subject matter easier and more efficient. In addition, mathematicians strive to improve algorithms to make them more compact, more efficient, and more general applicable. Given the importance of designing and using algorithms in today's society, learning to think algorithmically and understanding how algorithms work even gains in importance.

Views on mathematics education

The aim of our exposition on mathematical understanding was to clarify what we mean, when we advocate for shifting the emphasis from skills to understanding. In the foregoing, we gave two arguments for such a shift: (1) a one-sided emphasis on skills may lead to task propensity, resulting in isolated skills, which get mixed up easily; and (2) in the society of the future, calculating exceedingly becomes an activity of computers, and understanding mathematics becomes increasingly important for people working with computerized apparatus. We explicated that the type 2 understanding, which we believe is needed, transcends the type of understanding that is commonly aimed for (type 1). We added that a choice to aim for type 2 understanding cannot be separated from a choice for a certain form of mathematics education. Both result from a certain view of mathematics. Here, we may refer to Freudenthal (1991) who observes that algorithms can be taught, and concepts can be transmitted using language rich definitions. He rejects this *common way* of teaching mathematics, which connects this to a view in which mathematics is seen as a set of algorithms and definitions. He argues that mathematics should neither be taught as form, nor as content. Instead, the interplay between the two should do justice to mathematics as an activity. Byers (1999) also describes the former view, which he calls a logical-deductive perspective on mathematics, and he adds a second, ambiguous-metaphorical perspective. Byers follows the definition of Koestler: "Ambiguity involves a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference" (Koestler, 1964, cited by Byers, 1999, p. 28). He argues that mathematics is not merely algorithmic and deductive, and shows how important breakthroughs in mathematics involve overcoming ambiguity, paradox, and contradiction. He further claims that when you think of mathematics in only a logical-deductive way, you will never fully be able to understand the subtlety of mathematics and, more importantly, to fully understand the power of mathematics.

Focusing on isolated tasks using a pedagogy with a logical-deductive perspective will not result in type 2 understanding. But, more importantly, it may even hinder reaching this understanding, for example, by blocking the mathematical attitude of questioning things, or by trying to deny the ambiguity in the mathematics in instruction. We argue that allowing children room for questioning, reasoning, and nuancing does not harm the goal of mastering skills or mathematical techniques, on the contrary. In the other way around, types of pedagogies in which students have to follow examples and in which there is little or no room for questioning and reasoning may undermine or block type 2 understanding.

Reasoning with numbers as objects

In the foregoing, we have shown that there is a need for more conceptual goals, because (a) teaching isolated skills is not effective, and (b) the digital society asks for mathematical understanding. Further, we have argued that the kind of understanding that is needed is deep conceptual understanding (type 2). The deep mathematical understanding we are aiming for is broad in character, with mathematical objects at its core. In the following, we will discuss what this means for mathematics education in the domain of number. As an introductory remark, we may note that a focus on mathematical objects does not ask for a change in the content of the common basic skills. As far as a change in this respect is in order, it is motivated by the thought that calculations with larger and more complex numbers will be delegated to apparatus. An additional remark we want to make is that our position is that fostering process-object transitions is not just a goal in and of itself. What we also deem important is that numbers as objects allow for flexibly using number relations to solve tasks in the domain of number. We will first elucidate what the process-object duality entails in the domain of number. In doing so, we will link numbers-as-objects to Van Hiele's (1973) notion of junctions in networks of number relations. Next, we will work out how decomposing and (re)composing numbers-as-objects allows for the flexible use of number relations – offering an alternative for the use of a series of procedures. In the following, we will take addition and subtraction of whole numbers – up to 20, respectively up to 100 – and fractions and fraction operations as examples. First, however, we will discuss the notion of mathematical objects.

Process-object

As we have been rather implicit on what we understand by *mathematical objects*, we will elaborate on this notion now. In doing so, we will start with early number.

At a certain age, young children do not yet understand the question: "How much is 4+4?". Even though, they may very well understand that 4 apples and 4 apples together is 8 apples. The explanation for this phenomenon is that, for them, number is still tied to countable objects, like in "One, two, three, four. That is four apples". At a higher level, 4 will be associated with various number relations, such as:

$$4 = 2 + 2 = 3 + 1 = 5 - 1 = 8 \div 2, \text{ etc.}$$

At this higher level, numbers have become object-like entities that derive their meaning form a network of number relations (c.f. Van Hiele, 1973). We denote these object-like entities *mathematical objects*.

At the base of this development lies the activity of counting. Resultative counting results in the emergence of numbers as quantities ("four apples"). Next, the students engage in combining, comparing, separating, or completing quantities of marbles, blocks, or apples, etc. In this manner, they identify number relations in which numbers emerge as mathematical objects. To borrow Sfard's (1991) terminology, we speak of processes (counting) being turned into objects (numbers). This does not mean, however, that the new mathematical objects are separated from their genesis. In fact, students need to see what Sfard (1991) calls a *structural conception* – involving mathematical objects – and an *operational conception* – which concerns processes, algorithms, and actions as two sides of the same coin. According to her, learning mathematics mirrors the history of mathematics that shows a pattern of mathematical processes being turned into objects, which, in turn, are used in new processes[4].

Addition and subtraction of whole numbers

First step: derived facts

When students have networks of number relations at their disposal, they can solve various tasks with so-called *derived-facts strategies*. Although, we may note that the denotation *strategies*, as it is used in relation to addition and subtraction, for instance, is a misnomer. *Strategies*, such as *bridging ten*, *using doubles*, etc., are, in fact, procedures. The only aspect that might be called strategic is the choice of procedure (Verschaffel, Luwel, Torbeyns, & van Dooren, 2009). However, if the students have formed networks of number relations, we may expect them to combine number relations, annex number facts, to derive the sought number relation – instead of following a procedure. When solving 7 + 8, for instance, various number relations may come to mind:

$$7 + 3 = 10;\ 7 + 7 = 14;\ 7 = 5 + 2;\ 8 = 7 + 1;\ 8 = 5 + 3;\ \ldots$$

When such relations are ready-to-hand to the students, they may use these relations as puzzle pieces that can be combined to find the answer (Figure 3).

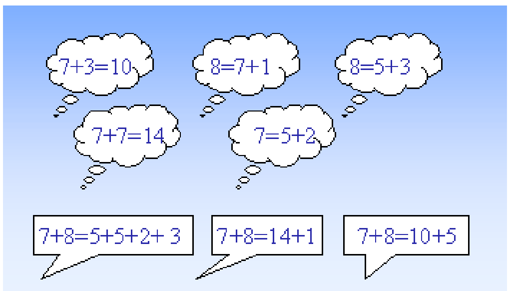


Figure 3. Solving 7 + 8

They may make combinations such as:

$$7 + 8 = 5 + 2 + 5 + 3 = 10 + 2 + 3 = 15,$$

$$7 + 8 = 7 + 7 + 1 = 14 + 1 = 15,$$
$$7 + 8 = 7 + 3 + 5 = 10 + 5 = 15.$$

To the observer it might seem that the students are using known procedures, but in fact, the students are decomposing and (re)composing numbers. Thus, we may conclude that by giving priority to understanding, by way of emphasizing the need to develop numbers as mathematical objects – annex junctions in networks of number relations – we also offer students a basis for solving addition and subtraction tasks in a variety of ways.

Next step: larger numbers

Students may solve addition and subtraction problems up to 100 in a similar manner, by using number relations that are ready-to-hand to them, that is, by looking at number relations as if they are pieces of a puzzle, and trying to find combinations that can be used to solve the problems at hand. Number relations that are ready-to-hand may concern number relations involving decomposing and (re)composing numbers-as-objects, on the basis of the decimal structure, and number relations, that may be developed in analogy to those in the domain up to 20 (e.g., 2+7 versus 20+70).

In principle, derived-fact solutions could also work for numbers up to 1000, if the students have sufficient flexibility in composing and decomposing numbers up to 1000, and in coordinating units of one, ten, and one hundred in the context of the decimal position system. At the same time, because of the emphasis on understanding the addition and subtraction in relation to the properties of number as object, students may develop knowledge of how large number arithmetic can be performed in a more systematic way (like the column algorithms). Construing numbers as objects, and decomposing and (re)composing numbers-as-objects to form new objects, offers a conceptual basis for understanding those algorithms.

Fractions and fraction operations

We will explore this idea further for the domain of fractions. Fractions may be perceived as more complex in their ambiguity than natural numbers. Among others, because of their broad phenomenological basis, which are often denoted sub-constructs (Behr, Wachsmuth, Post, & Lesh, 1984; Kieren, 1980; Streefland, 1991), part-whole, measure, quotient, ratio number and multiplicative operator. This means that if fraction is fully developed, students can seamlessly switch between these perspectives. This implies that students have to get acquainted with all these perspectives. Similar to addition and subtraction, students develop both a network of fraction relations and an understanding of more general ideas. By combining $\frac{3}{4} = 3 \text{ x } \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, and $\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$, the relation $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$ can come to the fore as a derived fact, where, $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3 \text{ x } \frac{1}{4}$, can be seen as a number relation. However, the latter also represents a more general insight that holds for all (proper) fractions, that is the idea that $\frac{1}{4}$ can be seen as a unit. Likewise, expressions as $\frac{2}{7}+\frac{2}{7}+\frac{2}{7}=\frac{6}{7}$ are examples of multiplication as repeated addition. Another link to a general insight is offered by $\frac{3}{4}=1-\frac{1}{4}$ or $\frac{3}{5}=1-\frac{2}{5}$. Both instances are number relations, but they also represent the more general idea of *complement*. So, when these number relations are reflected upon, they may also lead to a kind of class of number relations ($\frac{\mathrm{p}}{\mathrm{n}}=1-\frac{(\mathrm{n}-\mathrm{p})}{\mathrm{n}}$, in informal terms).

For fraction becoming an object, several process-object transitions are made. Eventually all the sub-constructs are inextricably connected into that idea of fraction as an object, as is the case with a fraction as a division and the result of that division. For example, 3 ÷ 4 can initially be seen as the process of dividing, which has to be connected to the number $\frac{3}{4}$, which is, at that moment, already an object. Later on, both $\frac{3}{4}$ and 3 ÷ 4 have dual sides as process as well as object.

Another example can be found in the equivalence of fractions, a known difficulty in fraction learning: $\frac{3}{4}=\frac{6}{8}=\frac{9}{12}=\frac{12}{16}$ can be seen as instances of a network of number relationships. Again, understanding the equivalence itself in combination with the known network of number relations for whole numbers creates a kind of class of number relations. This understanding of equivalence entails much more than the common procedure for deriving an equivalent fraction (multiplying numerator and denominator with the same factor). There are more number relations that can be recognized, such as $\frac{3}{4}$ and $\frac{6}{8}$ representing divisions with the same outcome and that all numerators are $\frac{3}{4}$ of their denominator (for $\frac{12}{16}$ it holds that $\frac{3}{4} \times 16 = 12$). In all these examples, the insight is needed that fraction is both a process and an object.

To conclude, we want to look at multiplication and division with fractions. The starting point is the ambiguity of a fraction both as the division and as the outcome of that division. So, the fraction already has the division inside itself. How does that combine with an operation as multiplying by $\frac{3}{4}$ and dividing by $\frac{3}{4}$? On basis of their understanding of fractions students may come to see that the operation "$\times \frac{3}{4}$" can be decomposed in two operations: "× 3" and "÷ 4", and vice versa. Relations between operations can be explored with arrow language as a model (Figure 4).

$$\ldots \xrightarrow{\times 3} \ldots \xrightarrow{\div 4} \ldots \qquad \ldots \xrightarrow{\div 4} \ldots \xrightarrow{\times 3} \ldots$$

$$\ldots \xrightarrow{\quad \times \frac{3}{4} \quad} \ldots \qquad \ldots \xrightarrow{\quad \times \frac{3}{4} \quad} \ldots$$

Figure 4. Relations between the operations multiplication, division and fractions

Exploring relations between operations, students may find that,

- "3 x (2 ÷ 5)" corresponds to "(3 x 2) ÷ 5", or that

- "$\frac{2}{5} \div 4$" corresponds to "$\frac{2}{(5\times 4)}$".

These findings may be combined to reason that $\frac{2}{5} = 5 \times \frac{3}{4} = (2 \times 3) \div (5 \times 4)$. Once being fluent with decomposing and (re)composing operations students will be able to derive the rule for multiplying fractions by "multiplying the nominators and the denominators". In this sense, developing mathematical objects not only facilitates flexibly solving tasks involving multiplication and division of fractions, it also offers a basis for a deep understanding of multiplying fractions. Moreover, these insights will also lay a foundation for similar ways of reasoning in algebra.

Supporting students in developing numbers as objects

To complete our discussion of number skills and numbers as mathematical objects, we will discuss what instruction, aiming at developing numbers as objects and process-object transitions within addition and subtraction up to 100, might look like. Note, however, that we limit ourselves to the core, which will not do justice to the characterization of doing mathematics we gave earlier.

We start by noting that research shows that students spontaneously develop both string methods and splitting methods (Beishuizen, 1993). In light of the research of Kraemer (2011), we think it is wise to support students in coming to grips with both methods to avoid that they start working with own inventions that are not well founded. However, in both cases, the number relations the students use are basically limited to splits in tens and ones. Therefore, a next step is needed, in which the students expand the range of number relations involved in conceptualizing 2-digit numbers as objects. Moreover, both methods have a procedural character. To come to see addition and subtraction as objects, the students have to transcend the dichotomy between splitting and jumping.

In the following we will describe the ingredients of a course with which these goals could be reached, which consist of three sequences. The basis is formed by two instructional sequences that are the results of design-research projects: one that supports students in splitting tens and ones, known as the Candy Shop (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997), and one that supports the string method, building on measurement (Stephan, Bowers, Cobb, & Gravemeijer, 2004). The third sequence – which is only tentative at this moment – aims at both extending the number relations that are formed (and used) in the first two sequences, and also fosters the transition from jumping and splitting as procedures to construing sums and differences as objects. These three sequences might be integrated in a larger sequence in the aforementioned order. In the following, we will discuss these sequences one by one, however, without trying to integrate them at this stage.

Candy Shop

The first sequence we may build on is the Candy Shop – which is developed by Cobb and Yackel. We will briefly sketch the sequence as it is described in Cobb et al. (1997). The sequence centers on a Candy Shop, where both loose candies and rolls of ten candies are sold. In this context, regularly candies have to be packed into rolls of ten candies, or rolls have to be unpacked. Initially, the students work with Unifix cubes that signify candies. These are stacked in stacks of ten in order to enact packing candies in rolls and so forth (Figure 5).

Figure 5. Single Unifix cubes and stacks of ten

The process of packing 10 candies into a role of ten results in the roll as the product of this process, and, at the same time, as an object that the students can use and manipulate.

In the next phase, the students are introduced to using pictured collections of candies to solve tasks, which involve generating alternative partitionings by composing and decomposing units of ten by packing and unpacking rolls of candies (Figure 6).

These pictured collections are described verbally, and recorded with "r" and "p" as abbreviations, e. g., 3r 4p, 34p, etc. On the surface, these pictured collections are just substitutes for the Unifix cubes signifying candies. However, there is a change in meaning; from signifying individual or groups of candies to signifying (numerical) quantities. Moreover, the function of the symbolizations changes. First, the meaning of the candies/Unifix cubes is situated in the story of the Candy Shop. Later on, when the students use pictured collections to show alternative partitionings, the concerns and interests of the students are primarily mathematical. The rolls then instantiate tens as mathematical objects, and individual candies instantiate abstract units of one. In relation to this, we may speak of a model-of/model-for shift (Gravemeijer, 1999): the tactile symbolizations initially function as models of the packing/unpacking activity. Later, the pictured collections function as models for reasoning about numbers up to 100 as mathematical objects that can be composed and decomposed in various ways.

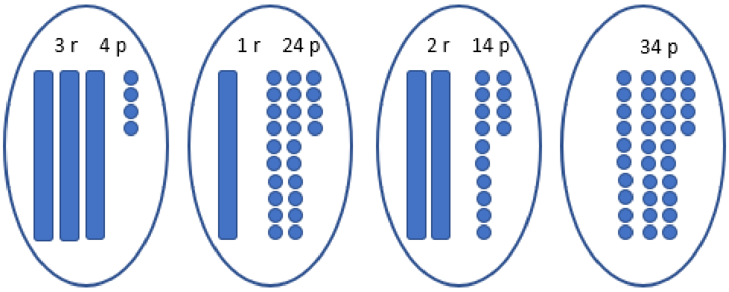


Figure 6. Pictured collections showing alternative partitionings where "r" and "p" are abbreviations of roll(s) and piece(s), respectively

Linear measurement and the number line

As a second source of inspiration, we describe the measurement/number-line sequence developed by Stephan, Bowers, Cobb, and Gravemeijer (2004). This sequence aims at supporting the string method. Here, the ambiguity of the numbers on a ruler – indicating both a position and a magnitude – may be exploited to foster the coordination of the cardinal and ordinal aspects of numbers, while the jumps used in adding and subtracting are symbolized with arcs on an empty number line. Grounding the string method in measuring fits with Freudenthal's (1983) phenomenological analysis of addition and subtraction. He advised to introduce a number line, in the context of measurement, as a means of symbolizing addition and subtraction.

After some introductory activities, the teacher tells a story about Smurfs – little blue dwarfs – who measure with food cans. These food cans happen to have the size of Unifix cubes. In the context of the story, the students start measuring lengths by iterating Unifix cubes[5]. As iterating with a unit of one becomes tedious, a bigger unit is asked for. After some discussion, this leads to the activity of measuring with both a bar of ten Unifix cubes stacked together and individual Unifix cubes. For practical reasons, the bar and cubes are subsequently replaced by a paper strip of ten on which the individual units are marked. Later on, 10 strips are glued together to create a ruler of 100, which is used as a tool for measuring.

While solving problems (which not only concern measuring lengths, but also involve incrementing, decrementing and comparing lengths), the activity of measuring with bars and cubes develops into coordinating units of 10 and 1. Furthermore, the students start using arithmetical solution methods to solve problems on incrementing, decrementing and comparing lengths – while the ruler is used as a means of support, for instance, when solving problems such as the following (Figure 7).

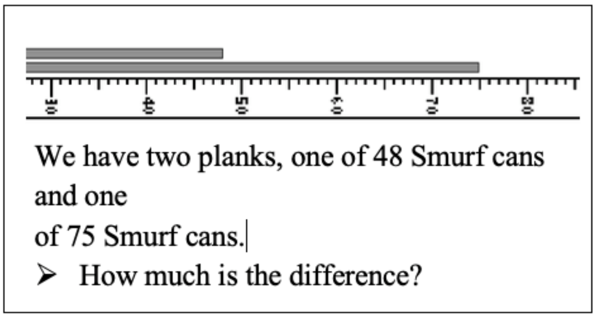


Figure 7. Comparing lengths

Here, students may look at the ruler and reason like this: 48+2=50; 50+10=60; 60+10=70; 70+5=75, thus the difference is 2+10+10+5=27. Such solutions procedures can be modeled with an empty number line (Figure 8).

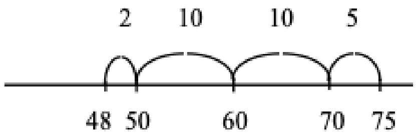


Figure 8. Counting by jumps on the number line

This use of the number line can be extended to all sorts of addition and subtraction problems.

In a later phase, the empty number line may be used to depict more sophisticated strategies. Looking at the numbers in the above problem, a student might think of 75 – 50 = 25 as a nice familiar number relation. This student might recast the problem in terms of a subtraction task: 75 – 48 = …, which could be solved via 75 – 50 = 25; 25 + 2 = 27. When justifying his or her strategy, this student might use the number line to show that "minus 48" equals "minus 50 plus 2" (Figure 9).

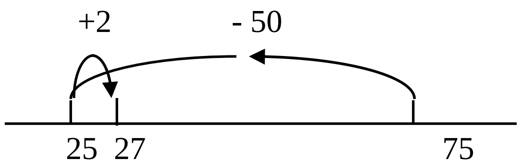


Figure 9. Compensating

In relation to this, we may speak of the model-of/model-for shift (Gravemeijer, 1999). Initially, measuring with the ruler functions as a model of iterating units of ten and one. Later, the jumps on the empty number line function as models for more formal reasoning about number relations. In this process, a shift is taking place in what the numbers signify for the students, which is similar to the shift that takes place in the Candy Shop context.

Here, numbers first refer to distances. Later, numbers start to signify mathematical objects that can be composed and decomposed in various ways. This sequence allows for the use of a wider variety of number relations, as the example of, 75 – 48 = …, via 75 – 50 = 25; 25 + 2 = 27 shows. The sequence, however, does not foster the development of number relations such as 75 = 50 + 25. Moreover, the jumping method has a rather procedural character too.

Transcending the splitting/jumping dichotomy

As we argued above, a follow-up annex extension is needed. We have to expand the set of number relations that the students have ready to hand, and we have to move away from the procedural character of splitting and jumping. By supporting the students in expanding the variety of number relations, they will also be supported in transcending the splitting/jumping dichotomy. Building on Van Hiele (1973), we may argue that expanding the variety of number relations will help students to come to see (2-digit) numbers as junctions in networks of number relations. Consequently, they will start to solve addition and subtraction problems purely by reasoning with number relations. In other words, they will come to think of adding and subtracting as composing and decomposing mathematical objects. As an additional design heuristic, we may follow the reasoning of Sfard (1991), that the reification of processes may be fostered by putting the students in a position where they are required to reason with processes as if they are objects already.

In our view, these number relations should include multiplication and division as well, where multiplication may be grounded in grouping, jumping and repeated addition.

In designing such activities, we might try to think of a model as means of support for a transition from processes to objects. Tasks such as the one shown in Figure 10 might be used as starting points in a model-of/model-for transition. The task asks for making a food box of a given price. The solutions could be notated as in Figure 11.

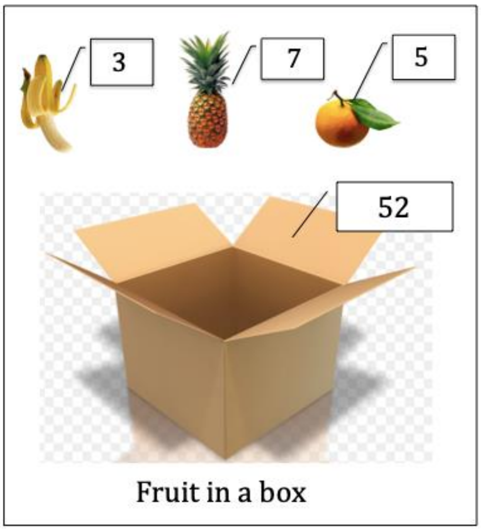


Figure 10. Packing fruit in a box, which will cost 52 euros

```
                                                                  4x3
                                                                  /
   7               7               28 = 4x7        32 = 4x8
  /               /               /               /       \
52              52              52              52           4x5
  \               \               \               \
   45 = 9x5        45 = 15x3       24 = 8x3        20 = 4x5
```

Figure 11. Solutions

This notation can be used for both situations, which can be thought of as splitting, and as combining (Figure 12).

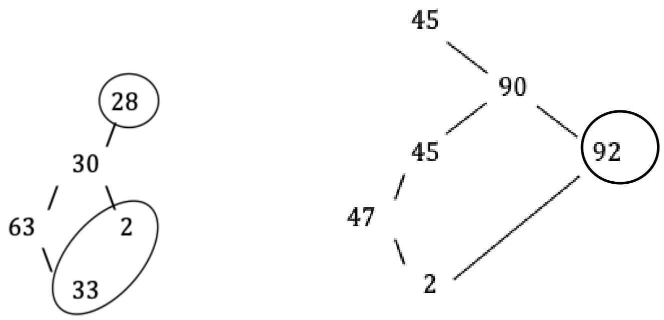


Figure 12. Splitting (63 – 28), on the left, and combining (47 + 45), on the right

In combination with this, tasks may be presented that create the need for treating sums and differences as objects, e.g., tasks such as,

- (77 – 35) – (44 – 35) =…, which may be solved by comparing the objects "77 – 35" and "44 – 35" and reason that "77 – 35" is 33 more than "44 – 35";
- 95 – 49 = …, which may be solved by comparing "95 – 49" and "95 – 50", knowing that 95 – 50 = … is easier to solve. Reasoning that 95 – 50 = 45, and thus 95 – 49 = 46.

Eventually, ambiguous visualizations, which signify both how a number can be split, and how that number can be composed from these components (Figure 13), may be used to discuss all possible interpretations of one composition/decomposition as one connected whole, which can simultaneously be thought of as 63 = 27 + 36, or 63 - 27 = 36, or 63 - 36 = 27, or 27 + 36 = 63, etc.

```
   27
  /
63
  \
   36
```

Figure 13. Composition/decomposition of 63

The schematic representation then comes to the fore as a model for reasoning about relations between sums and differences, while numbers become mental objects that can be composed and decomposed in different ways. At the same time, sums and differences become mental objects in and of themselves that are equivalent to the corresponding numbers. These relations, in turn, form the basis for flexible arithmetic and allow for expansion to bigger numbers, integers, and algebra.

In reflection, we believe that the example of the instructional design in the domain of addition and subtraction up to 100 can be construed as a paradigm case that can be informative, and allows for designing sequences aiming at the construction of mathematical objects in other domains. In this respect, we may point to the underlying principle of constructing or reinventing mathematics. We may further point to emergent modeling as a design heuristic. Those design principles – together with the guideline of transforming processes into objects by creating situations in which students are expected to treat processes as if they are objects already – may be put to use when designing sequences for other topics.

Conclusion

We started this article by reporting that Dutch research shows that teaching isolated skills is not effective. It results in low proficiency, low flexibility, and little understanding. We also observed that the digital society of the 21st century requires a high level of mathematical understanding, while the decreasing demand for routine skills allows for directing teaching time from teaching routine skills to fostering mathematical understanding. We elaborated what mathematical understanding to aim for. We identified several general types of goals to clarify what we consider *deep understanding*. We pointed to the duality of mathematical concepts – encompassing both an operational (process) and structural conception (object) – as a central objective for education. We elaborated on this idea by discussing what reasoning with numbers as objects implies for various topics. For the domain of basic skills, we noticed that this for one boils down to composing and decomposing mathematical objects. Flexible use of number relations – which are part of the constitution of numbers as objects – might be considered an alternative for standard procedures provided that the number range is limited. More complex tasks with larger numbers may be solved with machines in the future. At the same time, this flexible use of number relations may foster a deeper understanding of the operations at hand, which also offers a basis for understanding the standard algorithms for larger numbers.

We further discussed three instructional sequences, which together could support students in constructing both numbers up to 100, and sums and differences, as mathematical objects. Hereto, students have to transcend the dichotomy between string methods, and splitting methods – which both are operational in character.

We conjecture that the example of addition and subtraction up to 100 offers a paradigm case that can be informative for designing sequences aiming at the construction of mathematical objects in other domains. Given the importance of deep mathematical understanding in the digital society, we argue for a further exploration of a possible shift in this direction.

Notes

[1] Kraemer (2011) distinguished three groups of students, weak, average, and strong. The percentages represent the averages for, respectively, the weaker and stronger students.
[2] Steffe and Cobb (2012, p. 297) use the expression, "ten as a numerical composite", "to characterize a student's view of ten as a composite of ten ones".
[3] A mathematical attitude, in our view, mirrors the characteristics we mentioned above, striving for generality, exactness, brevity, and certainty.
[4] According to Sfard (1991), this transition encompasses three phases, *interiorization*, *condensation*, and *reification*. She depicts the latter as a sudden shift, which does not seem to fit the domain of number where reification involves a gradual expansion of networks of number relations.
[5] In the original teaching experiment it was taken-as-shared in the classroom that the cubical part of a Unifix cube signified a food can without the protrusion. To avoid confusion, one might instead want to use wooden cubes.

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