Identifying some key characteristics of an integrated approach to teaching modelling

Identificação das principais caraterísticas de uma abordagem integrada ao ensino da modelação

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Abstract. Proponents of an integrated approach to teaching mathematical modelling recognize that mathematical modelling and applications must be integrated into and contribute to elementary and secondary students' overall mathematical education. This study seeks to identify the chief characteristics of an integrated approach to teaching modelling by examining a selection of papers published as part of the proceedings of the International Conferences on the Teaching of Mathematical Modelling and Applications beginning in the 1990s, particularly the ICMI 14 study. Two perspectives are identified, each underpinned by a different purpose. One purpose is to solve a real-world problem, where the direction is from the real world to a mathematical world. The second purpose is to deepen students' understanding of developed representations in a mathematical world, where the direction is from a mathematical world to the real world. This study discusses the rationale for an integrated approach to teaching modelling under four main headings: its significance, key ideas for developing an appropriate primary and junior secondary curriculum, implications for classroom teaching, and the need for further research.
Keywords: modelling; integrated approach; mathematical knowledge construction; curriculum; teaching.

Resumo. Os proponentes de uma abordagem integrada da modelação matemática no ensino defendem que a modelação matemática e as aplicações devem ser integradas e contribuir para a educação matemática geral dos alunos do ensino básico e secundário. Este estudo procura identificar as principais características de uma abordagem integrada ao ensino da modelação, examinando uma seleção de artigos publicados como parte das atas das *International Conferences on the Teaching of*

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Mathematical Modelling and Applications, que se iniciaram na década de 1990, particularmente o estudo ICMI 14. Duas perspetivas são identificadas, cada uma sustentada por um propósito diferente. Um dos propósitos é resolver um problema do mundo real, em que a direção é do mundo real para um mundo matemático. O segundo propósito é aprofundar a compreensão dos alunos sobre as representações desenvolvidas num mundo matemático, em que a direção é de um mundo matemático para o mundo real. Este estudo discute a fundamentação de uma abordagem integrada ao ensino da modelação, sob quatro tópicos principais: o seu significado, as ideias-chave para o desenvolvimento de um currículo adequado para o ensino básico e secundário, as implicações para a prática de sala de aula e a necessidade de estudos adicionais.
Palavras-chave: modelação; abordagem integrada; construção de conhecimento matemático; currículo; ensino.

Introduction

The teaching of and learning about mathematical modelling have been justified from two main perspectives: one argues that students should engage in modelling to *develop competency in applying mathematics and building mathematical models*; the other advocates the use of applications and modelling in *learning about mathematics* (Niss, Blum, & Galbraith, 2007). Some authors describe modelling simply as a content or set of behaviors or competencies to be learned, in contrast to modelling as a vehicle (Julie & Mudaly, 2007). Modelling competencies have been researched and discussed by analysing modelling processes from several points of view (Kaiser, 2015). However, much less discussion has considered the linking of modelling to mathematical knowledge construction. This linking can be seen in particular mathematical programs designed for elementary and junior high school students, such as the University of Chicago School Mathematics Project (UCSMP; Usiskin, 1989, 1991), Realistic Mathematics Education (RME; Gravemeijer, 1993, 2007), and Model-Eliciting Activities (MEA; Lesh, 2003; Lesh & Yoon, 2007). Blum and Niss (1989) were the first to discuss an integrated teaching approach to modelling, emphasizing the constructing of mathematical knowledge through modelling. Since then, a vigorous discussion has continued about the challenges inherent in promoting integrated modelling approaches among authors such as Blum (1991), Swan (1991), Ponte (1993), and Garfunkel (1993). Blum (2015) and Blomhøj (2019) recently focused on how to include mathematical models in mathematics teaching at various educational levels, and how to justify the teaching of modelling and applications at these levels.

This study seeks to identify the key ideas associated with the development of an integrated modelling approach. We focus on a selection of conference papers published by the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) as well as the ICMI 14 Study, which outlines the key elements of, and the justification for, an integrated modelling approach.

The research contributions on which this paper is based have been published almost entirely by ICTMA in its biennial conference proceedings. It would have been possible to look at other sources, but ICTMA is unique in having focussed consistently on the teaching of mathematical modelling and applications. The ICTMA community, of which we have been active participants, has recognised and endeavoured to answer a key curriculum question; namely, that mathematical modelling, while readily justified when studied at university in science and engineering courses, needs a different justification for its inclusion in school mathematics, especially in the primary and middle school years. This is the chief reason why this study focuses on ICTMA-sponsored publications. Future research might well look at sources published in non-ICTMA series.

Key ideas associated with the development of an integrated modelling approach have been addressed regularly by contributors to ICTMA publications with shared meanings and a sense of continuity in exploring the issue, for example, by cross-referencing to what has gone before. The papers we have selected represent a range of answers to that key curriculum question. They provide different perspectives in terms of their theoretical assumptions and their international diversity allows us to be confident that our analysis, while not exhaustive, is comprehensive and informative to readers of this special issue of Quadrante.

The papers considered in this chapter focus on elementary and junior high school levels. These have been analysed based on four research questions: (1) What significance do they have for an integrated modelling approach? (2) What key ideas do they present in developing the curriculum at elementary and junior high school levels based on an integrated modelling approach? (3) What key ideas do they present in developing classroom teaching based on an integrated modelling approach, and how do these characteristics of classroom activities help define an integrated modelling approach? (4) What further research questions need to be tackled about the integrated modelling approach?

The study summarizes the results of analyses of these questions, which are examined in the order shown above, thus highlighting their essential points and their diversity.

Significance of an integrated modelling approach

This section seeks to identify the significant features of an integrated modelling approach to teaching from a mathematical point of view. Here, we discuss two perspectives on this question and emphasize the constructing of mathematical knowledge through modelling.

Emphasizing mathematising

The first perspective is the idea that *mathematising* and *constructing mathematical knowledge* are embedded in a series of mathematical activities. Gravemeijer (1993) took these key ideas from Freudenthal (1973, 1991) and defined mathematics as an activity within which mathematising plays a central role. Freudenthal argued that, in developing mathematics, formal mathematical knowledge constitutes the end point of the process of mathematising. In the top–down approach, these end points are taken as starting points, an approach that qualifies as an anti-didactical inversion. The fundamental principle is that formal mathematical knowledge is gradually constructed via repeated *mathematising*. The two elements of *mathematising* and *constructing mathematical knowledge* can be considered indispensable to mathematical activity.

A similar idea is advocated by Confrey and Maloney (2007), who base modelling on Dewey's description of *inquiry*. In their definition of modelling, Confrey and Maloney (2007) underline the interconnection between modelling and mathematical knowledge: "The modelling produces an outcome—a model—which is a description or a representation of the situation, drawn from the mathematical disciplines, in relation to the person's experience, which itself has changed through the modelling process" (p. 60).

This discussion brings us back to the two contrasting views on the purposes of modelling—one arguing that it merely solves a real-world problem and the other positing that it incorporates the outcomes of that activity. These outcomes are not always mathematical knowledge; for example, they may be indicators useable in real-life situations. However, the outcomes can often be mathematical knowledge that is useful in itself as well as in the real world.

However, two challenges are inherent to this approach. The first is treating mathematising, which involves complexity in the real world, and the second is treating the construction of mathematical knowledge that might initially be represented informally but later becomes more formal. Confrey (2007) asked "What epistemological ramifications result from intensified use of complexity?" (p. 126). Although complexity in the real world may appear to hinder students' development of new mathematical knowledge, it has several advantages. In realistic mathematics education (RME), for example, complexity in modelling tasks allows the teacher to foster formalization by generalizing solution procedures and models for other situations (Gravemeijer, 1993). In MEA, complexity—referring to the existence of several sets of data in the original problem—allows students to understand why other plausible ideas are less useful for a given situation (Lesh, 2003). Lesh contrasted traditional teaching with MEA and noted that

> some meanings depend as much on knowing why a given idea is not appropriate in some situations. However, the distinctions, connections, and assumptions that occur naturally in situations where students express, test, and revise their own ways of thinking about big ideas are seldom emphasized when instructors try to guide students along elegant paths to ideas they want to teach. (p. 50)

Emphasizing interpreting/contrasting

The second perspective is that modelling and applications can produce meaningful mathematical concepts by linking the mathematical model to real-world contexts. A mathematical model is based on a certain interpretation of reality (Matos & Carreira, 1997). Modelling and applications enable students to understand mathematical concepts by giving meaning to mathematics. From this perspective, linking mathematics to reality is more important than mathematising. The existence of two worlds, a real world and a mathematical world, makes it possible to interpret knowledge in a real-world context. Mathematical knowledge is not isolated but is strongly connected to the real world.

Carreira (2001) refers to the concept of *metaphor* (Lakoff & Johnson, 1980), which acts as the primordial element in the construction of models. Once in action, a metaphor provides the mediating structure between two domains. The modelling process has a metaphorical genesis, and the meanings of the metaphorical matrix are what make the model meaningful. In this sense, modelling is interpreted as the interplay between the real world and mathematics. Accordingly, Hanna and Janke (2007) and Ikeda and Stephens (2011) drew attention to an inverse modelling process, from a mathematical world into a real world. This will be discussed later.

Beyond the idea of connecting plural elements existing in different (i.e., real and mathematical) worlds, we should also consider the fact that contrasting two worlds may produce further questions. Hesse (1966) argued that

> the important thing about this kind of model-thinking in science is that there will generally be some properties of the model about which we do not yet know whether they are positive or negative analogies; these are the interesting properties, because, as I shall argue, they allow us to make new prediction. (p. 8)

Here, a model is used as a device to create new predictions or problems. Ikeda and Stephens (2015) suggested that a model in this context serves as a source of physical/mental entities that can be used in compare/contrast analyses.

Key ideas for developing elementary and junior high school curricula

Developing a mathematics curriculum requires attending to both the horizontal domain and to vertical sequences. One horizontal domain issue is how to *chunk* or classify, the contents of the mathematics curriculum. This is sometimes referred to as an issue of *grain size* and has sometimes been discussed in terms of segmenting the curriculum into topic areas. One vertical sequence issue is how to sequence the learning content within each domain according to a developmental sequence. In this section, we review the integrated modelling approach to the mathematics curriculum from the perspectives of these two issues.

Horizontal domain of school mathematics curriculum

In Japan, the horizontal domain concerns specific mathematical contents, such as numbers and formulas, geometric figures, function, and statistics. Modelling is intended to be taught in each mathematical domain. As a result, modelling is typically taught at the beginning or in the summary section of the unit. In this approach, students have already learned the mathematical skills that will be applied to solve real-world problems. By contrast, the Applications/Reforms in Secondary Education (ARISE) proposed by Garfunkel (1993) as an alternative design for producing an integrated curriculum for Years 9 to 11 is organized by modelling theme as opposed to content theme. In the ARISE design, nine domains are set as concept structures: fairness; codes; concepts of space; symmetry, pattern, and asymmetry; optimization; mathematical modelling; change and growth; risk; and conflict and competition. Mathematical skills were subsequently specified for each of these concept domains. Garfunkel (1993) proposed this concept structure to represent purposes or situations in which students can use and apply mathematics in the real world. For example, function as a mathematical skill is located within five of the above concept structures: fairness; symmetry, pattern, and asymmetry; optimization; mathematical modelling; and change and growth. In addition, concrete modelling and application problems are located within each concept structure. The ARISE project focuses on the concept structures concerned with the purposes and situations of mathematics use in the real world, as opposed to the traditional focus on mathematical content. This design is intended to construct mathematical knowledge through a focus on modelling. While the approach of the ARISE project is useful for developing a mathematics curriculum organized around different modelling themes, the challenge for teachers is how to connect the mathematical skills used in the concept structures and how to integrate key mathematical concepts overall.

Vertical sequences of a mathematics curriculum

Freudenthal (1983) stressed that multiple characterizations of mathematical concepts are required if their relationships with the real world are to be understood. Based on this assertion, Usiskin (1991) introduced the idea of *learning hierarchies*, which are concerned with the multiple meanings of mathematics concepts in the real world. For example, he lists six meanings for *number*: count, measure, location, ratio comparison, code, and derived constant. He also explains how to teach the graphical representation of velocity as the rate of change of distance comprising at least three contents: velocity as the rate of change of distance, representation of velocity, and representation of distance. Selecting velocity as the rate of change of distance requires an understanding of at least two key ideas: rate of change (slope) and velocity. Learning hierarchies are constructed in this way, and they play an important role in developing vertical sequences of integrated curricula. Usiskin (1991) argued that students are often unable to handle a process because they do not understand some elements of the hierarchy or have not been taught them.

Usiskin (1991) also pointed out that the modelling process can be described as an attempt to find mathematical concepts that are isomorphic to situations in the real world. The utility of the model depends on its degree of isomorphism (the closeness of its fit to real-world situations). Thus, he suggested that one should first consider models that are isomorphic and move gradually to models that are not, thus proceeding from the exact model to an almost-exact theory-based model, and then to an impressionistic model.

Lehrer and Schauble (2007) proposed a vertical sequence principle based on analogies. Analogies, they argued, are not mere copies, so testing is required to determine which aspects and relations of a more familiar system are pertinent for understanding the new system. From this perspective, one informational resource for pedagogical design is research that considers how analogical reasoning develops. This shift from literal similarity to mapping relations is a hallmark of analogical reasoning.

Key ideas for developing classroom teaching

In this section, we review several key characteristics of mathematical knowledge and their relevance to the integrated modelling approach. Then, we analyse the kinds of teaching processes that may be conducive to implementing it.

Essential features of mathematical knowledge

Considering the teaching of mathematical knowledge through modelling requires that we determine what is meant by *understanding a mathematical concept*. Usiskin (1991) identified four dimensions of mathematical understanding that are relevant to teaching and learning at the elementary and junior secondary levels:

(a) Skill-algorithm dimension: here, understanding is demonstrated by doing. Usiskin argued that this ranges from memorizing basic facts to carrying out procedures and inventing algorithms.

(b) The underpinning mathematical properties dimension: here, understanding is demonstrated by, for example, being able to name properties and their justifications; at the highest level, this involves the discovery of proofs.

(c) The use-application dimension: here, understanding is demonstrated by knowing when and how to apply an idea. This dimension ranges from straightforward one-step applications of mathematical principles to modelling.

(d) The representation-metaphor dimension: here, understanding is demonstrated by knowing why and being able to present or find metaphors. This dimension includes the use of concrete materials to represent mathematical properties and the creation of new metaphors or representations.

These four dimensions are necessary for clarifying the kinds of mathematical understanding that can be developed through modelling and for identifying how to characterize the relationships between modelling and the construction of mathematical knowledge. We will discuss the *use-application dimension*, which can be applied whenever students reflect on their own modelling activity and find situations in which existing mathematical knowledge can be applied.

When considering the character of mathematical knowledge, it is useful to refer to the dual nature of mathematical concepts. For example, Sfard (1991) asserted that this dual outlook can be achieved only by seeing a concept in both operational and structural terms. If we assume that an operational concept proceeds as a structural concept, the crucial question becomes how the reification of a process can be turned into an object. As Sfard (1991) noted:

> Here is a vicious circle: on one hand, without attempt at the higher-level interiorization, the reification will not occur; on the other hand, the existence of objects on which the higher-level processes are performed seems indispensable for the interiorization… the lower-level reification and the higher-level interiorization are prerequisite for each other! (p. 31)

Taking up Sfard's challenge, Niss (2013) asked "How can students learn to anticipate putting mathematical knowledge to work in modeling before they have learned modeling?" (p. 57). Niss applied Sfard's idea to teaching mathematical knowledge through modelling and added,

> In principle this leads to an infinite regression, leading to a learning paradox similar to that identified by Sfard (1991) in relation to the reification of process generated mathematical concepts—where an object resulting from reification of a process cannot be perceived as an object without considering it as being subjected to new process operating on it. (pp. 57-58)

This issue is discussed in the next subsection.

Challenges of implementing an integrated modelling approach

Swan (1991) pointed to the gaps or tensions between teachers' long-range intentions and students' immediate focus. The students may think that the problem *in hand* is the focus, whereas the teacher has a goal in mind that is several steps removed from the current problem and is related to a specific piece of mathematics that he or she wants the students to learn. While students are working on a particular project or problem, their main objective is to obtain the answer, not necessarily to develop particular mathematical techniques. Students often see mathematics as a tool and not as an end in itself. Students who are likely to deploy only those skills with which they are already confident may resist any attempts to teach them new techniques while they are involved in their projects. However, the teacher may wish to draw on opportunities offered in a given module to motivate students to learn specific mathematical techniques in a more explicit way. The pedagogical challenge is to balance teaching specific mathematical knowledge without destroying the essential flow of activities contained in a module. This issue is connected to the discussion above, which contrasted the activity of mathematising with the construction of significant mathematics knowledge.

Swan also analysed the advantages and disadvantages of teaching mathematical concepts before, during, and after students' problem-solving activities. If a teacher introduces mathematical concepts before students have engaged in problem-solving activities, these new techniques may seem artificial and disconnected because the students do not see the need for them. Students may also tend to assume that the module is merely a vehicle for practicing these techniques, rather than for developing autonomy in problem solving. Teachers who teach mathematical concepts during students' problem-solving activities can respond to their needs as they arise, but the work on the module can tend to drag on over many weeks and lead to boredom. Teachers who teach mathematical concepts after students' problem-solving activities may be able to motivate students to see the value of the techniques when they are taught them. However, students may still not be able to use these techniques autonomously unless they are given opportunities to apply them in other real-world problem-solving contexts. An integrated modelling approach recommends that new mathematical knowledge be taught, or at least consolidated, after the students have engaged in problem-solving activities.

Teaching approaches emphasizing mathematising

There are two related approaches to developing mathematical models as mathematical knowledge through mathematical modelling. The first is what might be called *emergent modelling*, following Gravemeijer (2007). The second is what might be called *model-eliciting approaches*, as described by Lesh and Yoon (2007). Both approaches emphasize moving from an *informal/implicit* model to a *formal/explicit* model, and both widen the applicable range for systemizing mathematical knowledge.

In emergent modelling, the long-term development of more abstract mathematical knowledge is focused on formalization and level-raising by generalizing solution procedures and applying models to other situations. Gravemeijer (2007) used several metaphors for this process, such as transitioning from horizontal to vertical mathematising and moving from a *model of* a situation into a *model for* mathematical reasoning. Addressing the reification problem, Gravemeijer (2007) claimed that

> the transition from a model of to a model for coincides with a progression from informal to more formal mathematical reasoning that is interwoven with the creation of some new mathematical reality—consisting of mathematical objects (Sfard, 1991) within a framework of mathematical relations. (p. 140)

Following the question posed by Niss (2013), we can offer an alternative interpretation of the transition from a *model of* to a *model for* in relation to reification by drawing attention to two sequential purposes in the transition from a *model* to a *model for*. The first purpose arises in the real world: solving a real-world problem. For this purpose, operational processes are developed. At this stage, it is not necessary to change the operational processes into objects. In the next stage, however, the purpose is changed to deepen students' mathematical understanding of developed representations, which can now be applied to different situations in the real world according to the *use-application dimension* identified by Usiskin (1991). The reification of a process into an object is not generated in the first purpose but is generated gradually once students start to be conscious of the second purpose.

As discussed by Lesh and Yoon (2007), we can see a similar process in model-eliciting activity, consisting of two sequential purpose stages. English (2003) argued that one of the key goals of mathematical modelling is the development of generalized conceptual systems for children. She claimed that a key criterion in designing modelling problems for children is that the tasks should have the potential to elicit mathematically significant constructs that ultimately become generalizable and reusable. However, if the highest priority is assigned to the diversity of mathematisation, there may be a case that does not lead to the construction of significant mathematical knowledge. Lesh and Yoon (2007) claimed that, "if the models involve mathematically significant concepts, then model development tends to involve significant forms of concept development" (p. 164). However, it also seems that long-term learning processes in emergent modelling are needed to target specific mathematical knowledge first and then develop real-world problems so that the solutions can lead to the intended mathematical knowledge. Gravemeijer (1993) gave two examples: two-digit addition/subtraction and data representation (Gravemeijer, 2007). Changing the order between mathematising and constructing significant mathematical knowledge should cause a difference between the two approaches. Relevant decisions about the appropriate order of priority might depend on the framework adopted in a particular mathematics curriculum. In Japan's mathematics curriculum, horizontal domains are created on the basis of classifying mathematical contents. The first order of priority is to construct significant mathematical knowledge and use this knowledge later for mathematising. By contrast, in the ARISE program, the order of priority is reversed, as discussed by Garfunkel (1993); the main purpose is mathematising and the secondary purpose is constructing significant mathematical knowledge to enable mathematising to proceed.

In a modified version of classical modelling, Lamon (1998) conducted long-term classroom activities consisting of the quantitative analysis of elementary situations at the beginning of the year, structured modelling in the middle of the year, and more open modelling activities at the end of the year. The quantitative analysis of elementary situations involved constructing mathematical knowledge through modelling. At this stage, quantitative relationships were represented in multiple ways: verbal statements, use of arrow notation, and graphs. Then, problems with the same structure were introduced, in which new categories were added or existing categories were refined as necessary. In time, students were encouraged to group together structurally similar relationships, and thus gradually support a vocabulary of conceptual terms that can be applied to describe algebraic phenomena. For example, one type of relationship can be described as *proportional*, and another can be described as *exponential*. While accumulating experience by analysing and describing real situations mathematically, students can be assisted in developing a sophisticated understanding of concepts such as rate of change; slope; and constant, average, and instantaneous change. This process seems to differ from emergent modelling. There are at least two sequential stages. One purpose arises in the real world; this is to solve a real-world problem, as a result of which a variety of quantitative analyses of elementary situations are developed as operational processes. In this stage, it is not necessary to change the operational processes into objects. In the next stage, the purpose is to deepen students' understanding of developed representations in a mathematical world.

Blomhøj and Kjeldsen (2013) described the reification of a process into an object as a crucial step in developing mathematical concepts at the university level. Implementing an integrated modelling approach for university students involves issues that differ from those involved in elementary and junior high school education. Blomhøj and Kjeldsen (2013) analysed the case of differential equations and pointed out students' difficulties in shifting from viewing a differential equation as a relation between the momentary rate of change to viewing it as a relation between a function and its derivative. In one example, the teacher explained that the growth rate was proportional to the square of the size of the population. Two purposes thus emerge: one related to modelling, such as "viewing a differential equation as a relation between the momentary rate of change and the actual size of a certain function (here the size of the population)" (p. 151); and the other related to a mathematical issue, such as viewing it as a relation between a function and its derivative. When these two sequential purpose stages are elicited at the same time, students have difficulty separating them.

Teaching approaches emphasizing interpreting/contrasting

Connecting plural elements existing in different worlds is crucial to forming the meanings of mathematical knowledge concerned with the representation–metaphor dimension identified by Usiskin (1991). As mentioned, the reverse modelling process, from a mathematical world into a real world, is discussed by Blum (1998), who introduced the concept of *reality-related proving*. This occurs in three steps: (1) *interpreting* the premises (certain mathematical objects or operations and certain interrelations) in a specific real context; (2) performing *arguments* or *actions* within this context through contextualized knowledge; and (3) *translating* these results *back* into mathematics and obtaining mathematical results. Blum (1998) asserted that this activity is not merely a temporary stage on the way to formal proofs but can be interpreted epistemologically as an appropriate means of revealing the meaning of certain mathematical facts.

Similarly, Hanna (2003) discussed the teaching of mathematical proof using physical principles that support the goal of not only proving that a mathematical proposition is true but also clearly showing why it is true. Hanna (1993) pointed out the difference between mathematical proof using physical principles and reality-related proving (Hanna & Janke, 2007): whereas reality-related proving may be taken to be informal, a proof using physical principles may enjoy the same degree of mathematical rigor as any other deductive proof.

Ikeda and Stephens (2011) analysed Japanese historical textbooks published during in 1941–42 and showed how a real-world situation can be used as evidence of how to expand number concepts—in other words, how to define the rule of the multiplication of negative numbers by using a real-world situation. While this approach may show how mathematical knowledge can be constructed by focusing on moving from a real-world situation into a mathematical model, attention should also be paid to the complementary activity of connecting mathematical knowledge to real-world situations. This reverse approach supports the development of mathematical concepts by validating them in concretized models. This approach is similar to reality-related proving, as discussed by Blum (1998), which is more concerned with defining or proving.

The common purpose of these three activities is to deepen students' understanding of representations in a mathematical world. Assumptions (element 1) and conclusions (element 2) are the two key elements of the mathematical world. Blum (1998) and Hanna (2003) both provide assumptions and conclusions wherein students consider contextual reasoning by using real-world situations. By contrast, Ikeda and Stephens (2011) show only assumptions to students, wherein they are invited to find the conclusions through contextual reasoning by using real-world situations. By considering the difference between the two cases, we can see the alternative situation where no element of mathematical world is shown to students. In this situation, students elicit assumptions and conclusions by solving a real-world problem and then connecting assumptions to conclusions as mathematical properties. For example, when considering probability y of settling paper-scissors-rocks with x people, the following two models may be elicited:

$$Model\ 1\colon y = \left(\frac{1}{3}\right)^{x-1}(2^x - 2) \qquad Model\ 2\colon y = \left(\frac{1}{3}\right)^{x-1}\left\{\binom{x}{1} + \binom{x}{2} + \cdots + \binom{x}{x-1}\right\}$$

As the answer is unique, we can obtain the following representations:

$$2^x - 2 = \binom{x}{1} + \binom{x}{2} + \cdots + \binom{x}{x-1}$$

By interpreting the meaning of "2," it can be represented as follows. In this stage, the reverse direction from the mathematical formula to the meaning in a real-world situation is required:

$$2^x = \binom{x}{0} + \binom{x}{1} + \binom{x}{2} + \cdots + \binom{x}{x-1} + \binom{x}{x}$$

The above embodies a simple model of the binomial theorem that corresponds to the *underpinning mathematical properties dimension*, where understanding is demonstrated by, for example, being able to name properties and their justifications and, at the highest level, perform the discovery of proofs, as identified by Usiskin (1991). The challenge for teachers is to determine the feasibility and appropriateness of these kinds of activities as important contexts and opportunities for integrating modelling activity and generating mathematical knowledge.

Teaching in an integrated modelling approach

In this section, we examine several of the essential features of the teaching process described above to characterize an integrated modelling approach. The long-term goal of having students develop more abstract mathematical knowledge in emergent modelling has two sequential purposes. The first and most obvious purpose is to solve a real-world problem in which students are required to move from the real world into a mathematical world. During this stage, several operational processes are developed. The second purpose is to widen the range of situations so that the developed operational processes can be applied and possibly extended. The underlying purpose of this phase is to deepen students' understanding of developed representations in a mathematical world. In this second phase, the direction is from the mathematical world to the real world. We can see the same two sequential purpose stages in Lamon (1998), although her processes seem to be different from emergent modelling. The first purpose is to solve a real-world problem in which the direction is from the real world to a mathematical world. However, subsequent activities are intended to add new categories or refine existing ones. The underlying purpose is to deepen students' understanding of developed representations in a mathematical world. Real-world situations are used to check the validity of the initial structure and refine categories in a mathematical world. In emergent modelling, mathematics knowledge is developed by widening the range of situations so that the developed operational processes can be applied and possibly extended. By contrast, in Lamon's approach, mathematics knowledge is developed by clarifying the similarities and differences between the developed plural operational processes. The two approaches differ in their methods of constructing mathematical knowledge; however, they have key similarities. In particular, students do not construct mathematical knowledge by focusing only on the direction from the real world into a mathematical world; the reverse direction from the mathematical world to the real world can be as crucial as, and complement, the initial modelling activity.

For Blum (1998), Hanna (2003), and Ikeda and Stephens (2011), the preferred direction is from the mathematical world to the real world. Their common purpose is to deepen students' understanding of the mathematical world of the representations they have developed; this reveals the meanings in the real world, which correspond to mathematical propositions. However, since mathematical propositions do not fall from heaven, we need to ask about the purposes that underpin mathematical propositions. Therefore, it might be useful to interpret the existence of at least two sequential stages: the first purpose is derived from a real world or a mathematical world, and the second purpose is derived from a mathematical world.

From these considerations, we can describe the integrated modelling approach as two complementary activities composed of two sequential purposes. The first purpose is to solve a problem that has occurred in the real world or in a mathematical world; the second purpose is to deepen students' understanding of the developed representations in a mathematical world, through which the operational processes become refined and systematized as mathematical objects. These objects can then be applied to a wide range of situations exhibiting similar structures or new categories, revealing the significance of the developed mathematical representation. These characteristics of an integrated modelling approach should be refined by examining other cases.

Other challenging issues

In this section, we address several other issues that need to be considered regarding the teaching of mathematical knowledge through modelling.

As Matos (1998) noted, the words used in a mathematics book or by the teacher to describe a situation to be modelled are important mediating elements when students try to make sense of the situation. Borromeo Ferri and Lesh (2013) noted that the situation model has been used in connection with non-complex modelling problems, specifically word problems. The wording used in a mathematics book can be understood as a world element that differs from the real world. By contrast, Gravemeijer (1993) accepts informal schemas and notations as mathematical models because they will become mathematical models after generalizing and formalizing processes, even though this model is called a *real model* by Blum (1993) because it is so close to reality. For example, Borromeo Ferri and Lesh (2013) distinguished between implicit and explicit models, arguing that implicit models are often held at an unconscious level, whereas explicit models are more conscious and can be communicated to others more effectively. Examining the authors reviewed in this study reveals several paired words denoting opposites, such as *informal* and *formal* models, *implicit* models and *explicit* models, *real* models and *mathematical* models, and *situation* models and *mathematical* models. These distinctions reflect different viewpoints: from a mathematical viewpoint, the distinction may be between ungeneralized and generalized; from a psychological viewpoint, it may be between unconscious and conscious; and, from a linguistic viewpoint, it may be between non-communicable and communicable.

Ikeda and Stephens (2017) discussed modelling as an interactive translation between plural worlds, which are not simply the result of arbitrarily changing mathematical representations but arise fundamentally through comparisons and contradictions between competing perspectives. The principle underpinning this framework is that mathematics can be abstracted repeatedly from one world to another and in both directions. From the perspective of plural worlds, the intention is not only to promote the development of modelling competency but also to deepen students' mathematical knowledge by connecting and integrating the outcomes constructed in each world (Ikeda & Stephens, 2020).

Summary

We analysed the characteristics of the integrated approach to teaching modelling by focusing on a selection of papers published by ICTMA and ICMI 14, starting from the 1990s. Our aim was to elucidate the various perspectives that have been advanced to support an integrated approach to teaching modelling and applications as part of school mathematics.

Two perspectives on the significance of the integrated modelling approach were identified. The first argues that mathematical activities are predominant. These activities are composed of two fundamental elements: mathematising and constructing mathematical knowledge. This might be described as a *traditional* and essentially one-directional perspective, in which modelling produces outcomes that represent the situation being modelled. The outcomes can be used as mathematical knowledge in both the mathematics domain and the real world. In the second, contrasting viewpoint, modelling can be represented as an interplay between a real word and a mathematical world. This perspective is a multi-directional one, in which linking mathematics to reality is a more important issue than mathematising. By contrasting the two worlds' elements, it becomes possible to interpret and deepen the meaning of mathematical knowledge.

In this multi-directional approach, multiple activities can be considered, such as applying the developed representation to a wide range of situations, seeking the same structures or new categories among developed multiple mathematical representations, and revealing the meaning of the developed mathematical representation.

Attending to both the horizontal and vertical domains is fundamental to developing an integrated modelling approach. We discussed Garfunkel's idea of *concept structure* (Garfunkel, 1993) and contrasted this with Usiskin's ideas of *learning hierarchies* and *degrees of isomorphism* (Usiskin, 1991). Finally, we presented the idea of *stages of analogical reasoning* advanced by Lehrer and Schauble (2007). These different but complementary ways of thinking directly address the key curriculum question we outlined at the start of this chapter: how an integrated modelling approach can contribute to students' overall mathematical understanding and support sound curriculum development?

In our survey of the papers considered in this paper, several epistemological perspectives rest beneath the surface. We were not able to give these contrasting perspectives and dichotomies—such as the *informal* versus *formal*, *implicit* versus *explicit*, *real* versus *mathematical*, and *situational* versus *mathematical* pairings—the attention they deserve. These pairings reflect multiple viewpoints based on mathematical, psychological, didactical, and linguistic traditions. These categories are not presented as an exclusive list but to remind us of the importance of identifying the various epistemological and didactical perspectives that lie beneath the surface when we discuss the importance of any integrated approach to teaching mathematical modelling in schools.

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