Epidemiological models and the problem of coherence: from the critical justification to a practice of teaching mathematical modelling

Modelos epidemiológicos e o problema da coerência: entre a justificação crítica e uma prática de ensino da modelação matemática

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Abstract. We explore the critical justification for mathematical modelling in school as one that aims at enabling students to reflect – internally and externally – on the role of mathematical models in describing and shaping risk phenomena, determining people's courses of action, and informing actual policy decision-making. We argue for the exemplarity of epidemiology from three perspectives: subjective, instrumental, and critical. Respectively, these refer to the case being exemplary for students' experienced lives, mathematical ideas and competencies, and the formatting power of mathematics in society. By analysing epidemic modelling workshops with high school students in Denmark, we claim that it is possible to live up to this justification, though some challenges remain. Possibilities arise by invoking real scenarios as a departure point and scaffolding the modelling process in a dialogical setting. The main challenge is to balance the scaffolding and prompting with dialogical features in a limited timeframe.
Keywords: critical justification; epidemics; exemplarity; high school; mathematical modelling; SIR model.

Resumo. Exploramos a justificação crítica para a modelação matemática na escola como aquela que permite aos alunos refletir – interna e externamente – sobre o papel dos modelos matemáticos na descrição e modelação dos fenómenos de risco, determinando os cursos de ação das pessoas e informando a tomada de decisão política. Discutimos a exemplaridade da epidemiologia a partir de três perspetivas; subjetiva, instrumental e crítica. Respetivamente, estas referem-se ao caso de ser

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exemplar para a experiência de vida dos alunos, ideias e competências matemáticas e o poder de formatação da matemática na sociedade. Ao analisar workshops de modelação de epidemias com alunos do ensino secundário na Dinamarca, afirmamos que é possível cumprir essa justificativa, embora alguns desafios permaneçam. As possibilidades surgem ao invocar cenários reais como ponto de partida e estruturando o processo de modelação num ambiente dialógico. O principal desafio é equilibrar o andaime e as sugestões com recursos dialógicos num período de tempo limitado.
Palavras-chave: justificação crítica; epidemia; exemplaridade; ensino secundário; modelação matemática; modelo SIR.

Introduction

In 1992, the sociologist Ulrich Beck published a framework to understand our contemporary society as a world risk society, a new paradigm to address the lost illusion of control of modernity (Beck, 1992). Risk phenomena include climate change, instalment of nuclear power plants, diplomacy and armed conflicts, management of global financial economics, and infectious diseases such as the so-called mad cow disease, Ebola, and COVID-19. Together with corresponding responses from international organisations, national governments and the public, they provide illustrative cases for the theoretical underpinnings, politics and research in our risk society, as portrayed by Beck's (2000) own revisitation.

Though few economies have a disproportionate share in greenhouse gas emissions, the devastating consequences of climate change have been perceived all over the world, showing how "even the antithesis between global and local is short-circuited by risks" (Beck, 2000, p. 217). The debate on the use of nuclear power takes place years before plants are installed, illustrating that risks are not the same as destruction, rather the "perception of threatening risks determines thought and action" (p. 212) and thus "the past loses [to a threatening future] its power to determine the present" (p. 214). On the same line, international diplomacy acts primarily unbeknownst to the general public by people who foresee the risk of armed conflicts.

Additional to these aspects, the case of epidemics of infectious diseases also highlights the "loss of clear distinction between nature and culture" (Beck, 2000, p. 220), where the biological evolution of a virus' reach and harm is conditioned by personal and social behaviour. For example, other than treatment and vaccination programmes, the governmental responses to the COVID-19 pandemic include closing borders, social distancing recommendations, and stay-at-home rules.

All these phenomena can and have been modelled mathematically. Moreover, the use of mathematical models to inform governmental strategies stands out as a clear illustration of Skovsmose's (e.g., 2005) notion of the formatting power of mathematics. For example, the real-time rate of infection of a disease, often referred to as the reproduction rate, is a concept only existing in epidemiological mathematical models. Yet, its estimation determines the course of action; the tightening or loosening of restrictions. Mathematics, in that sense, is not only attempting to model and understand the world as it is, but it is also shaping it.

What is the role of mathematics education in this societal reality? Niss and Blum (2020) have identified two main overarching reasons for including mathematical modelling in school, which they call "modelling for the sake of mathematics" and "mathematics for the sake of modelling" (p. 28). In other words, mathematical modelling arises both as a means to incentivise students to learn and appreciate mathematics and as a mathematical-based practice that is to be learned in its own right. However, the justifications for teaching mathematical modelling in school need not be in line with teaching students to become – necessarily – mathematical modellers, since school mathematics is a different type of knowledge than that of professional modellers, i.e. "an educational process cannot be interpreted as a straightforward process of enculturation" (Alrø & Skovsmose, 2002, p. 255). What is common to all students is that they will become adult citizens, and "critiquing modelling is part of the learning that takes place in the process of doing modelling, and one of the aims is to produce critical, politically engaged citizens" (Barbosa, 2006, p. 296). This type of justification, referred to as the *critical justification* (Blum & Niss, 1991) or *socio-critical perspective* (Barbosa, 2006; Kaiser et al., 2006) of modelling and applications in school, calls for activities by which students can critically reflect on the societal use and formatting power of mathematical models.

A coherent transformation from such general justification to teaching practice is not trivial. For once, experts rely on mathematical models whose complexity overwhelms the school mathematics curriculum. Simplifying the mathematics of these models to fit the curriculum may compromise the authenticity upon societal usage. Moreover, even if the mathematics is well suited for a certain school level, provoking critique is not necessarily achieved by solving a mathematical task put in context. As Alrø and Skovsmose (2002) put it, "a process of learning critically cannot be a casual or a forced activity. It is not possible to prescribe that anybody should be critical" (p. 231). Thereby, we are taking these developmental challenges as a point of departure for a theoretical discussion. First, we argue that epidemiology is an exemplary area for teaching mathematical modelling under a critical justification. Second, we analyse a modelling workshop in epidemiology for talented high school students intended to be coherent with the critical justification for mathematical modelling. From here, we address the research question: *What are the possibilities and challenges connected to the teaching of modelling in epidemiology under the critical perspective on mathematical modelling in school?*

We begin by clarifying what *exemplarity* means as a pedagogical concept. We can give precision to the claim that epidemiology is exemplary in addressing what it can be exemplary for. Moving towards the critical justification, we follow by arguing for consistent approaches to teaching practices that enable *critique* as an observable construct in mathematical modelling in school. We illustrate these discussion points by describing and analysing an extra-curricular workshop on mathematical modelling of epidemics for high school students held by the first author. The article is not meant to be a report on a case study, and thus presents itself not as an empirical contribution but as a theoretical contribution, which uses the case study as an exemplary source for a general discussion of the role of mathematical modelling for understanding risk phenomena and of what it can mean to live up to the critical justification for teaching mathematical modelling.

The exemplary principle

In their definition of the critical competence argument, Blum and Niss (1991) mention that such competence should enable students to assess “representative *examples* of actual uses of mathematics, including (suggested) solutions to socially significant problems” (p. 43, emphasis added). As a didactical consequence, a teaching practice coherent with such an argument must begin using appropriate examples.

As a pedagogical concept, the exemplary principle can be traced back to Martin Wagenschein. He argued for the use of in-depth studies of few well-chosen cases in institutionalised education, in such a way that “the partial serves as a mirror of the whole” (Wagenschein, 1956, p. 133f). Though some scholars interpret his viewpoint as a pragmatic “method of reducing curricula” (Andersen & Kjeldsen, 2015, p. 25), the exemplary principle refers not only to the mere use of cases in the classroom or to their specificity. According to Elicer (2020), exemplarity is, instead, their quality, i.e., the answer to the question of what is the case an example for, and we can identify at least three perspectives.

The first tradition is anchored in John Dewey’s notion of experiential learning for general pedagogy. Illeris (2002) argues that this perspective is oriented towards the learner’s *subjectivity*, and it deems a good problem as one that connects to students’ personal experiences. From here, the personal relevance to a student defines the quality of an example because “an experience arouses curiosity, strengthens initiative, and sets up desires and purposes that are sufficiently intense to carry a person over dead places in the future” (Dewey, 1938/2015, in Illeris, 2002, p. 148). Hereby, exemplarity serves to address students’ motivation to bear with the challenges of the learning process.

The second perspective refers to an example being *instrumental* for the disciplinary questions, concepts and methods, casting light over a whole field. As a science educator, Wagenschein was concerned about the limited transmission of an increasing body of knowledge. A good teaching practice ought to give students insight into the elementary problems he claimed all disciplines have. Concerning the teaching of mathematical modelling and applications, the instrumental perspective on exemplarity resonates with what Niss and Blum (2020, p. 28) label as “modelling for the sake of mathematics”, i.e. the use of applied cases as anchoring points to learn and appreciate mathematics. In addition, cases of mathematical modelling can be instrumental for understanding the interdisciplinary interplay between modelling and the sciences disciplines in which it is taking place.

Finally, the *critical* perspective on exemplarity focuses on the case as being exemplary for power structures in society. Moreover, it orients this awareness towards change and empowerment. This tradition can be traced back to the German philosopher and sociologist Oskar Negt, who, as an educator for the working class, redefined the exemplary principle as a pedagogical tool to provide “possibilities for self-liberation” (Negt, 1975, p. 42). Negt draws on C. Wright Mill’s notion of *sociological imagination*, defined as the ability to switch “from one point of view to another, (…) from the political to the psychological, from the examination of the individual family to the evaluation of the state budget and to realise structural connections between individual life stories, immediate interests, wishes, hopes and historical events” (Negt, 1975, p. 45). From a critical perspective, the quality of an example lies in its ability to illustrate the formatting power of mathematics. In his book *Towards a Philosophy of Critical Mathematics Education*, Skovsmose (1994) introduced this term as the paths leading from thinking abstractions to realised abstractions. These practices crystallise and partake of the normalised functioning of an increasingly automated society.

In sum, a teaching practice coherent with the critical justification calls for exemplary cases of actual uses of mathematical models in society. The selection of such examples must address what the case is exemplary for, which can be approached from three perspectives: subjective, instrumental, and critical. In what follows, we argue that mathematical modelling in epidemiology is exemplary in all three senses.

Exemplarity of epidemiology for teaching mathematical modelling

Mathematical modelling has played a crucial role in the epidemiology of infectious diseases for long (Bailey, 1986). Models are used to define concepts and notions, describe and explain data and phenomena, predict courses of epidemic infections, design and test vaccination programs or other types of regulations, and provide a basis for designing health care policies. Thus, epidemiology is a scientific discipline with a close interdisciplinary interplay with mathematical modelling in its foundation and societal application.

In addition, it is exemplary for an area in which applications of mathematical models play essential roles in political decision-making related to the handling of societal risk phenomena. That was true already before COVID-19, with illustrative cases such as the returning influenza epidemics and pandemics, the SARS epidemic, the design of vaccination programs against childhood diseases, the international campaign against copper, and the HIV pandemic. However, the COVID-19 pandemic has displayed the crucial role of mathematical modelling in controlling an infectious disease – at least in countries like

Denmark – to a degree never seen before. Thus, COVID-19 has amplified the pedagogical justifications related to the subjectivity and critical perspective of exemplarity. However, the main point here is to argue that mathematical modelling in epidemiology is exemplary in terms of all the three traditions of exemplarity mentioned above, even disregarding the COVID-19 pandemic.

The realisation of the pedagogical justifications related to exemplarity relies on the degree to which the teaching enables the students to experience the different perspectives of exemplarity. We will discuss it with the case presented later in the paper.

In the *subjectivity perspective on exemplarity*, an obvious point of departure is the fact that all upper secondary students have personal experiences with infectious diseases such as influenza.

Presenting and discussing with students a typical course of a viral infectious disease, which typically causes long-lasting immunity, as is the case with influenza, is a fitting way to activate the students' experiences (see Figure 1).

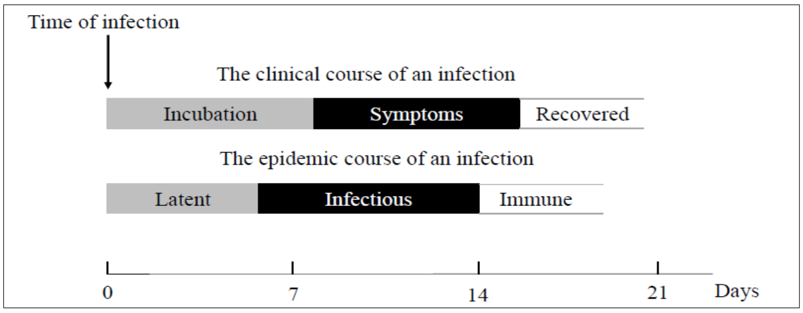

Figure 1. A typical course of a viral infection (Andreasen, 1995, p. 5)

Distinguishing between the clinical and the epidemic course of an infectious disease can qualify the students' personal experiences and establish a common starting point for a subsequent modelling process. The students can engage in discussing the typical behaviour during an influenza season.

After a few days of increasing symptoms, infected individuals may stay at home – in bed – and will therefore not infect as many as if they were able to continue their everyday life. The average time for being infectious effectively may therefore be deduced from Figure 1 as the time of incubation minus the time of latency plus maybe one or two days. For influenza, that is typically 3-5 days.

In addition, the subjectivity perspective includes students' knowledge about vaccinations programs against influenza, childhood diseases, and, of course, the vaccination campaign against COVID-19. Thus, the subjectivity perspective has students' experiences related to both a personal and a societal domain.

In the *instrumental perspective on exemplarity*, the area must enable the students to engage in modelling processes, contributing to their development of modelling competency

(Blomhøj & Jensen, 2007). In epidemiology, upper secondary students can understand the problem areas, which is the point of departure for its modelling. It is possible for them to follow and reflect upon – for example – the SIR model for infectious diseases. Moreover, in this work, they can be introduced to compartment modelling of dynamical systems. This approach is a solid and general technic for modelling dynamical systems traditionally leading to systems of ordinary differential equations. However, such mathematical systems are generally not accessible for analyses at the upper secondary level. However, as illustrated in the case analysed later, it is possible to mathematise compartment diagrams through systems of difference equations, which students can analyse numerically in a spreadsheet. With this, they can work with the modelling of more complex systems already at the upper secondary level, and at the same time develop their conceptual understanding of dynamical systems in ways that support their possible subsequent learning of differential equations.

In the *critical perspective of exemplarity*, the important thing is that the students gain experiences with different aspects of critique concerning mathematical modelling and the societal use of models in epidemiology. One form of critique is related to decisions taken in the modelling process. Here, the students can discuss different assumptions behind the SIR model and the estimation of parameter values in it. Another form of critique is related to the use of a mathematical model in a societal decision-making process. Epidemiology is rich in the use of mathematical models and model results to support healthcare policies. This relevant type of critique is very much depending on the purpose of using model results. The use of models in the handling of COVID-19 in Denmark is a clear illustration hereof. Relatively simple models were used to characterise the current developments in the number of infected by means of the reproduction number and to communicate to the public possible outcomes depending on different scenarios for restrictions on social interactions. A more complex model was developed to predict the prevalence in disaggregated age groups, and model results were used to support the decision of reopening society. Although one cannot get grabs of an exact basis for the political choices, it is possible to discuss with students already at the upper secondary level that such societal use of different mathematical models relies on their validity.

We must note that the activation of the pedagogical justifications based on exemplarity depends on the fact that students experience it. Exemplarity is primarily a potential of the example. The realisation of its exemplarity rests on the quality of the teaching.

Critique and mathematical modelling

The term *critical* refers to at least three meanings: the statement of *critical* or negative remarks, the response to a *crisis* or critical point, and *critique* as the unveiling of political, historical and other hidden meanings. All of them share the same etymological root in the

Greek word *krinein*, which refers to separating, judging, and deciding (Ernest, 2010; Skovsmose, 1994). Accordingly, and beyond selecting appropriate examples, a teaching practice coherent with the critical justification should allow critique to occur. The learning environment must present opportunities for students to make separations, judgements, and decisions about the role of mathematical models in society.

To portray different possibilities in the classroom, Alrø and Skovsmose (2002) offer a simple two-dimensional model to characterise milieus of learning. On the one dimension, the content may refer to mathematics, a semi-reality or real-life. On the other, these references can be approached through sequences of exercises or landscapes of investigation. Skovsmose (2011) advocates that, to enable critique, one should move towards the quadrant of real-life landscapes of investigation, where "uncertainties emerge. (…) However, a risk zone is a zone for possibilities" (p. 48). From here, there exists a broad umbrella of frameworks for action.

Among others, one can mention the problem-solving tradition, focused on developing competence and habits of mind by confronting students with non-routine and challenging problems (Schoenfeld, 1992). The realistic mathematics education programme (RME), in turn, emphasises the use of experientially real situations as a departure point for mathematisation (Gravemeijer, 1999), aligning itself to Dewey's educational philosophy. Another relevant approach is inquiry-based pedagogy in mathematics, loosely defined as having learners work as scientists or mathematicians do (Artigue & Blomhøj, 2013), having an up-front investigation statement and institutionalising mathematical knowledge at the end of the journey.

These and similar practices would, in principle, have students participating more actively in the construction of knowledge through a concrete applied and non-routine mathematical application. By doing so, the approach would activate its dialogical and collaborative dimensions (Alrø & Skovsmose, 2002; Artigue & Blomhøj, 2013). These interactions between students and the teacher are the basis for the unit of analysis: reflections. A reflection is

> a deliberate act of thinking about some actual or potential action aiming at understanding or improving the action. Reflections take place in the minds of individuals but are strongly influenced by social interactions, and they can only be detected and analysed through communicative acts. (Blomhøj & Kjeldsen, 2011, p. 386).

The object of students' reflections ought to determine whether they are critical.

Observing critique

Checking and *critiquing* are portrayed as cognitive skills on the revised Bloom's taxonomy (RBT), belonging to the broader category of *evaluating*. While checking is defined as "testing for internal inconsistencies or fallacies in an operation or act" (Anderson et al., 2001, p. 83), *critiquing* involves "judging a product or operation based on externally imposed criteria and standards" (p. 84). Anderson et al. (2001) draw the line between judging whether conclusions follow a scientist's method and whether the method is appropriate to solve the problem. These notions resonate with the distinction between internal and external reflections (Blomhøj, 2020; Blomhøj & Kjeldsen, 2011), where students may check or critique different steps within the mathematical modelling cycle. In other words, students' internal reflections refer to checking the assumptions, construction, and outcomes of a mathematical model. External reflections tackle the issue of becoming aware and judge the suitability of the mathematical model to real-life usage and its possible consequences.

From Barbosa's (2006) socio-critical perspective, different levels of critique are illustrated by students' discussions in doing mathematical modelling, which can be of the mathematical, technological, and reflective type. Skovsmose's (1992) entry points on reflective knowing offer a basis for concretisation. Mathematical (internal) discussions take form in questions of the kind "are our calculations right?" and "are we doing the right calculations?" Technological (internal) discussions refer to the reliability of the results for the purpose in mind. Reflective (external) discussions ask for the necessity of mathematics at all to solve the problem, the moulding of our conceptions of the world based on the model, about the possible use of the model and consequences hereof; and about our reflection process.

Skovsmose (1998) argues that *mathemacy*, a parallel competence to the notion of *literacy* (Freire & Macedo, 1987), should be understood as an epistemic unit where internal and external reflections are coordinated. Therefore, a mathematical modelling teaching practice coherent with the critical justification aims to allow and promote students' connected reflections in, with and about the use of the model.

The possibilities and challenges of teaching modelling of epidemics

Our empirical basis consists of experiences from a three-hour workshop on the modelling of epidemics for talented high school students from *gymnasium* grades 2 and 3 (age 16-18 years) following the A-level mathematics course (highest level) in Denmark. The workshop was offered by Roskilde University to the Academy for Talented Young Students (ATU, *Akademiet for Talentfulde Unge*) as part of a programme with extra-curricular activities at universities offered to selected – by their teachers – high school students showing particular interests in and talent for one or several academic subjects. In 2015-2019, the first author taught the workshop ten times for groups of 12-16 students with distinctive aptitude in mathematics.

We draw on the experiences from these workshops regarding the modelling work conducted, students' results and reflections during their work in small groups and the common discussions taking place at the workshop. In particular, we focus on those aspects that have been stable over the workshops to a high degree. The dialogues given for illustration are not verbatim transcripts but typical dialogues with the students, which, to a large extend, have proven to be reproducible in the workshops.

The experience from the workshops is analysed retrospectively with regard to the research question. To answer the research question, in the analysis of the workshops, we focus in particular on the three perspectives of exemplarity and pinpoint potentials and challenges regarding the development of the students' modelling competency within the area of epidemiology.

Setting the scene for the students' modelling activities

In Denmark, modelling is part of the mathematics curriculum; all the students have some experience analysing simple models and interpreting model results in a given context. Typically, they have seen (a simplified) model of a mathematical modelling process (Blomhøj & Jensen, 2007). However, only a few have experience with doing the modelling process themselves, and if so, they typically refer to experiences from modelling in physics. Moreover, only students having A-level in biology or biological technology have learned about human virus infections more thoroughly, and those students are rare exceptions at the workshops.

Consequently, the workshop begins with an introduction to some basic notions of infectious diseases, emphasising viral infections causing long-lasting immunity, such as influenza. This framing includes the typical course of virus infections from a clinical and epidemic perspective, as shown in Figure 1. The form is dialogical, and the students are invited to share their knowledge and personal experiences. The students attending the workshop come from different schools. Therefore, the introduction also serves to create a friendly atmosphere for dialogues with and among the students. In terms of exemplarity, the introduction primarily functions as a basis for establishing the students' subjectivity regarding the general problem of modelling the spread of an infectious disease. In addition, at the beginning of the workshop, it is emphasised that epidemiology is exemplary for a close interplay between a field of natural science and mathematical modelling.

To establish a basis for the students' engagement in the modelling process, they are introduced to the basic ideas behind the *SIR model*. That is the division of the population into *Susceptible, Infectious and Recovered* (immune) according to infectious status and that the number of individuals in each of these groups can be represented as functions, $S(t)$, $I(t)$, and $R(t)$. This representation includes the compartment diagram in Figure 2 and, in particular, the central mathematical idea behind compartment modelling: the change of each compartment over a short time interval can be determined as the sum of inflows minus the sum of outflows times the time interval. These ideas are thoroughly discussed with the students. In particular, the generality of the compartment approach to the modelling of dynamical phenomena is emphasized for the students. The students can only experience this aspect of the instrumental exemplarity through their subsequent use of the compartment approach in other contexts.

Compartment modelling leads naturally to systems of differential equations. All students have had differential calculus before the workshop, but they have not yet learned about differential equations. Therefore, at the workshop, the students were supported to mathematise the compartment model with a system of difference equations and to analyse the model numerically using a spreadsheet.

The students are presented with the challenge of modelling the dynamics of an epidemic caused by an infectious virus in a closed model population of 1000 individuals. The task is to develop a model as simple as possible, which captures the fundamental dynamics of an epidemic. The model should describe the one-directional flow of individuals through the phases of infection as indicated in Figure 1 and consider the infectiousness of the virus and the average time of being infectious. The task is communicated to the students by employing the slide shown in Figure 2.

Task 1 (15 minutes of work in pairs)

Express the flow between $S(t)$ and $I(t)$ as well as between $I(t)$ and $R(t)$ by means of these functions and the two parameters c and v based the assumptions discussed.

In small time interval Δt (where $c\ \Delta t << 1$) the changes in each compartments can be calculated for time $t + \Delta t$.

Complete the difference equations:

$S(t+ \Delta t) = S(t)$ - ...

$I(t+ \Delta t) = I(t) +$ ....

$R(t+ \Delta t) = R(t) +$ ....

Figure 2. The compartment diagram for the SIR model

Further, the following description of the basic assumptions behind the SIR model is presented to and discussed with the students:

1. During an epidemic, there is a one-directional transmission of individuals: S → I → R.
2. Thus, there is no loss of immunity in the model.
3. The population size N is constant during the epidemic, thus $N=S(t)+I(t)+R(t)$ for all t.
4. This assumption means that the model considers a closed population without birth, death and migration.
5. All persons have the same behaviour and response concerning the infection at hand.
6. The infectious behaviour in the population is described by just one parameter, the contact rate, c [day^{-1}], which is the number of *effective contacts* per day that each person

in the population is involved in. An effective contact is a contact that leads to infection if it occurs between a susceptible and an infectious.

7. The transition from I to R is determined by a cure rate v [day^{-1}], which is the proportion of I that becomes healthy (or, more precise, immune) per day. $1/v$ corresponds to the average time of residence in compartment I.

The compartment diagram shown in Figure 2 is used to structure and support the students' mathematisation of the assumptions. Given the limited timeframe, pre-structuring the mathematisation by focusing on the flows between the compartments is an important and, for most students, necessary support. The students work in pairs for approximate 20 minutes with mathematising assumptions 4 and 5 and completing the difference equations as asked in task 1. Despite the scaffolding in terms of a careful introduction to the notation, detailed explanations of the assumptions orally and in written language, and the prompt given for developing the difference equations, it is not a straightforward task for most pairs. This step is per the research literature emphasising the mathematisation sub-process as one of the main challenges for students in doing mathematical modelling (Niss & Blum, 2020).

Therefore, at this stage, the teacher dialogues with students to help and support them without doing the mathematisation for them. Some pairs have not yet put anything on paper when approached. A typical discussion with them runs like this:

Teacher:	What are you thinking? What should go into a formula expressing the number of expected new infections during a short period?
Student 1:	It must be something with c and Δt, right.
Student 2:	We should also take into account the number of immunes; they cannot be infected.
Teacher:	That is right. Who is it that can cause a new infection to occur?
Student 2:	Those that are already infected – those in the I-compartment.
Teacher:	Yeah, but can they do it alone?
Student 1:	No, they have to meet with a susceptible person.
Teacher:	Right! In addition, you (Student 1) mentioned the contact rate, c, and the length of the time interval, Δt, as relevant for expressing the number of new infections. Can you calculate the number of contacts that the susceptibles, $S(t)$, have during a time interval Δt?
Student 1:	That must be something like c times $S(t)$ times Δt.
Teacher:	Yes, exactly, and then we should find out how many of the contacts we expect would lead to infection. Try to read the definition of the contact rate c again.

This discussion exemplifies internal reflections as it focuses on the mathematisation sub-process. In particular, their discussions are mathematical (e.g., Student 1 proposes to multiply c, $S(t)$ and Δt to get the number of contacts carried out by the *Susceptibles*) and technological (e.g., Student 2 refers to the infectious as those being already in the I-compartment). The flow between I and R is somewhat more manageable for students to mathematise. Although the number of individuals leaving compartment I in Δt is simply proportional to the cure rate, v, the number of infectious at time t, $I(t)$, and the length of the time interval, Δt, the students typically need to go back and reread assumption 5. After around 20 minutes, most pairs are able to mathematise the two flows asked for in task 1 and complete the difference equations. The model is summed up by the teacher while referring back to the assumptions and emphasising the principle of inflows minus outflows for determining the changes in each compartment over a discrete-time step Δt, as the basis for setting up the system coupled difference equations for the SIR model:

$$S(t+\Delta t) = S(t) - cS(t)\frac{I(t)}{N}\Delta t$$
$$I(t+\Delta t) = I(t) + cS(t)\frac{I(t)}{N}\Delta t - vI(t)\Delta t$$
$$R(t+\Delta t) = R(t) + vI(t)\Delta t$$

The students' work with mathematising the two flows is essential for developing their modelling competency. Moreover, it also helps students understand the basic assumptions more deeply and why they are relevant and even necessary. However, they are rough simplifications of the diversity in which real people react and respond to an infection. During the work and subsequent discussion, internal reflections about the assumptions and their consequences for the meaning and estimation of the parameters are evoked. For example, some students reflect on the meaning of the contact rate c and how that could be estimated. This parameter encompasses both a societal aspect of the population and a biological element of the infectiousness of the virus in question. In fact, the contact rate c is a case of an artificial parameter, which is constructed in the modelling process, and therefore cannot be estimated independent of the model.

It is different with the cure rate v, which – at least in principle – can be estimated independent of the model as 1/(the average time of being infectious). In both cases, the students reflect on the degree to which it makes sense to assume that these parameters have the same value for all people in the population. The necessity and rationale behind these assumptions are discussed with the students. We know that, in real life, they are not fulfilled, but to be able to mathematise the flows between the compartments, we need these simplifying assumptions. Moreover, in real epidemic situations, there is typically insufficient scientific knowledge or empirical data for making the model more complex and estimating parameters for specific subgroups in the population.

Students analysing the SIR model using a spreadsheet

Building on the system of coupled difference equations for the SIR model, the students are given the task of developing a spreadsheet in approximately 15 minutes to analyse the model numerically with given parameters and initial values. For this task, the students are provided with a pre-structured spreadsheet (Figure 3). The pairing is organised to have the necessary skills for coding and copying spreadsheet formulas with parameters. Most of the students are pretty skilful in Excel.

Working with transforming the difference equations into spreadsheet formulas challenges the students to read meaning into the equations again and focus on the different roles played by the parameters, the state functions, and their initial values.

Construct a spreadsheet using in Excel for calculating the development in t, S, I and R from your difference equations.

The SIR model				
Contact rate *c*	0,85			
Cure rate *v*	0,4			
Time interval Δ*t*	0,1			
Population size *N*	1000			
Degree of vaccination	0			
Initial values		999	1	0
	Time *t*	*S*(*t*)	*I*(*t*)	*R*(*t*)
Initial values	0	999,0	1,0	0,0
The first calcutaed row	0,1	998,9	1,0	0,0

Figure 3. Task 2 with the structure for a spreadsheet SIR model

The experience of copying the formulas for calculating the development of the state functions step by step supports the students' understanding of how the dynamics of the epidemic is modelled in their SIR model.

The students use their spreadsheet to simulate an epidemic with given parameters and initial values, as in Figure 3. They produce graphs for S, I and R as functions of t (Figure 4) and graphs of I as a function of S for different initial values (Figure 5). During this work, the students were challenged by the teacher to interpret and reflect on their results.

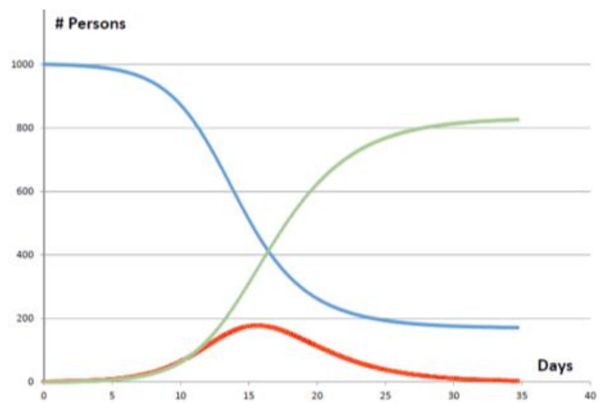

Figure 4. Solution curves, S(t) (blue), I(t) (red), R(t) (green) with S(0)= 998, I(0) = 1, R(0) = 0 and c = 0,85 and v = 0,4

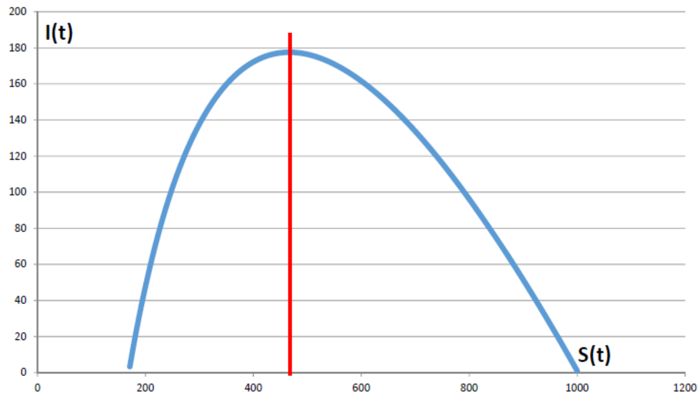

Figure 5. Solution curve, I(S), in the phase plane the same parameters and (S(0), I(0)) = (999,1)

As for the solution curves in Figure 4, a dialogue with a student pair runs like this:

Teacher:	What do you notice concerning how the epidemic develops in the model?
Student 1:	At first, the number of infected increases slowly, but after five days, it increases more rapidly until it reaches its maximum after approximate 16 days. Hereafter it decreases and becomes close to zero after 30 days.
Teacher:	Yeah, so the epidemic is all over in 30 days. What do you think of this?
Student 1:	It could be realistic for an influenza epidemic.
Teacher:	Did everyone in the population get infected in the model?
Student 2:	No, there are still nearly 200 susceptibles after 30 days.
Teacher:	And how is this in real life? Does the entire class get infected during an influenza epidemic?
Student 2:	No, not everybody.
Teacher:	Of course, during a real influenza season, we do not know if those who do not get infected were immune beforehand or just escaped the infection. However, your results show that in the model, it is possible to escape the infection as susceptible just by chance since the epidemic dies out before all susceptible individuals have been infected.

Internal reflections here refer to the sub-processes of the modelling cycle that come after the mathematical analysis. Namely, Student 1 begins interpreting the model results, and Student 2, prompted by the teacher, engages in an early validation of the model contrasted with perceived reality.

The last 30 minutes of the workshops are used for discussing the role of mathematical models in one or two of the four health care issues listed in Figure 6. Mathematical modelling is indispensable for understanding and for developing healthcare policies addressing all these issues. They have all been studied in students' modelling projects at the natural science bachelor programme at Roskilde University, employing modified and extended versions of the SIR model. In all cases, it is possible for the ATU students to understand and engage in external reflections related to using mathematical models as a basis for designing health care policies. For the second issue mentioned in Figure 6, this is discussed in Blomhøj (2020).

1. **How to optimize vaccination against influenza?**
   - which groups in the population should be offered vaccination for free?
   - how to ensure timely development of a vaccine effective against a current influenza virus?

2. **How to secure herd immunity for the so-called MMR childhood infections (measles, mumps and rubella)**
   - through information campaigns explaining herd immunity?
   - through legislation as in Germany?

3. **Should we install a vaccination program against HPV (Human papillomavirus infection) – causing cervical cancer**
   - is herd immunity relevant for that type of disease?
   - prevention through vaccination or through campaigns for safe sex?

4. **How to control the prevalence of sexual transmitted diseases such as gonorrhea and chlamydia**
   - through contact tracing or through preventive treatments to groups with high contact rates?

Figure 6. Four healthcare issues related to infectious diseases

Summing up findings from the workshops

To pave the way for answering the research question, we sum up the experiences from the workshops. We emphasise the particular conditions with relatively small groups of highly motivated and mathematically talented students and the intensive yet limited format in form of a three-hour workshop. These conditions are evidently different from ordinary mathematics teaching at high school. However, in Denmark as in many other countries, mathematical models and modelling is part of the curriculum, and it is even required to do some projects with the students. Therefore, despite the special conditions, we find that the experiences from the series of workshops can serve to answer the research question. In particular, we pinpoint the possibilities for the students to experience the three perspectives of exemplarity previously discussed.

As for the motivational aspect of the topic, which is connected to *the subjectivity perspective of exemplarity*, it is evident that the students attending the workshops find the area of mathematical modelling in epidemiology both interesting and relevant. They can relate it to personal experiences and their general knowledge about infectious diseases, and they can easily see the societal relevance of modelling in epidemiology. The workshop is thus exemplary in balancing personal motivation in experiential learning (e.g., Illeris, 2002), and social relevance, as defined by Wagenschein (1956). This was the case before COVID-

19. For sure, addressing the use of mathematical models in relation to the COVID-19 pandemic will increase the experienced relevance of mathematical modelling in epidemiology among high school students. The main didactical challenge connected to this potential is the orchestration of a whole class dialogue, which motivates the work, enables the students to share their experiences and reflections, and sets the scene for the students' modelling work. On the one hand, the workshops constitute an ideal situation in terms of motivation. On the other hand, the very limited time frame was a challenge.

The experience from the workshops shows that high school students can develop the SIR model in the form of a system of coupled difference equations from basic assumptions and analyse this model analytically and numerically using a spreadsheet. The students worked with all sub-processes in a mathematical modelling cycle, except for the problem formulation, and developed, in that sense, their mathematical modelling competency both in terms of the degree of coverage concerning the modelling cycle and in terms of the mathematical level of their modelling work (Blomhøj & Jensen, 2007).

Through subsequent work with mathematical modelling of dynamical phenomena in other contexts, the students can experience *the instrumental exemplarity* of the compartment approach. The primary didactical challenge is finding ways of supporting the students' modelling work without removing the learning potential. This challenge might be more complicated in a typical class situation than in the workshop due to the variation in the students' mathematical abilities. However, the limited time frame in the workshop made it necessary to scaffold the students' work carefully with the mathematisation process. That was done through the design of task 1. In their work with mathematising the compartment diagram given in the task (Figure 2), the students make sense of the assumptions as a necessary basis for representing each of the three groups, S, I and R, as a homogeneous compartment in which all individuals behave and react in the same way concerning the infection.

Moreover, in the mathematising process, the students can experience the interplay between the definitions of the involved parameters and the basic assumptions. The teacher pinpoints it in the dialogue shown in relation to task 1. In addition, the students' work with the mathematisation process is exemplary with regard to the instrumental use of basic and essential mathematical concepts such as variables, functions, proportionality, and the use of algebra in general.

The students experience the instrumental exemplarity in the compartment modelling of dynamical phenomena by means of systems of difference equations and numerical analysis in a spreadsheet developed by themselves. When the students later learn about differential equations in high school and some of them about systems of ordinary differential equations in further education, the chances are good that they can connect to their experiences with modelling infectious diseases with this experience.

In addition, through their work with analysing the model analytically, the students gain insights into concepts such as the reproduction number and rate, herd immunity and critical vaccination degree. These concepts are essential in epidemiology in general and – as witnessed by the COVID-19 pandemic – in real life. On a higher level of abstraction, the SIR model is exemplary for the interplay between modelling and the development of theoretical concepts and notions in the field of applied mathematical modelling.

As for *the critical perspective on exemplarity,* through the workshop, the students engage in internal reflections related to the modelling process and, to some extent, also in external reflections and discussions related to the use of mathematical models in healthcare issues related to infectious diseases (Blomhøj & Kjeldsen, 2011). As pinpointed in the analysis of the workshop, the internal reflections address the assumptions behind SIR. This type of reflection often takes place as an integrated part of the students' modelling work. Therefore, it is a crucial aspect of the teacher's interaction with the students to notice and support such reflections and make sure that they are shared and discussed in the class. Reflections that can lead to a critique of or alternatives to the model's basic assumptions are, of course, of particular relevance. For example, some of the students pinpoint that assumption 2 excludes the reality of death in the system for simplicity's sake. Therefore, the model is not valid for long term predictions or deadly infections.

External reflections related to the actual or possible societal use of mathematical models related to epidemiological health care issues are covered in the last part of the workshops. Here, the students are presented with four authentic healthcare issues (Figure 6), which can only be dealt with on rational grounds through mathematical modelling. At each workshop, one or two of these issues are discussed in some detail. Typically, half of the students draw on their internal reflections regarding the simple SIR model in these discussions. For example, concerning the first issue about the policy on influenza vaccination, some students questioned the ground on which an age-structured SIR model can be developed. Some students ask: Is it possible to estimate values of the contact rate for the different age groups? Concerning issues 2, 3 and 4, a rather intense discussion occurred, in some workshops, about the legitimacy of basing health care policies on uncertain mathematical models.

The workshops reveal the potential for engaging high school students in reflections about the role and function of mathematical models in societal health care issues related to infectious diseases and for connecting such discussions to the students' experiences with mathematical modelling in epidemiology.

Closing remarks

Looking at the world through the lens of the risk society allows rethinking the purpose of education in general and school mathematics in particular. We explore the critical justification for mathematical modelling in school as one that aims at enabling students to reflect on the role of mathematical models in describing and shaping risk phenomena, determining people's courses of action, and informing actual policy decision-making. We argue for the exemplarity of epidemiology from three perspectives: subjective, instrumental, and critical. By analysing epidemic modelling workshops with high school students in Denmark, we claim that it is possible to live up to this justification, though some challenges remain. A general message is that teaching qualities determine whether exemplarity is experienced.

For once, the topic provokes interest in students, as anyone can relate to the handling of infectious diseases. Nowadays, the COVID-19 outbreak, and its associated consequences and handling could be appealing to all students. However, the *subjectivity* perspective on exemplarity is not realised without the case being made of a point of departure for the modelling process. Making explicit connections between the matter and students' lives is consistent with inquiry-based teaching (Artigue & Blomhøj, 2013) and RME's call for making situations *experientially* real (Gravemeijer, 1999). Possibilities arise from having students doing the modelling themselves, while the challenge is to frame the activity in a dialogical fashion (Alrø & Skovsmose, 2002) in a limited timeframe.

Epidemiology is *instrumentally* exemplary for mathematical ideas of dynamical systems and developing a broader modelling competency (Blomhøj & Jensen, 2007). Aside from the problem formulation, students can go through all steps in the modelling cycle through a dialogical setting, both amongst the students and with the teacher. A pragmatic difficulty is taking the time to present and discuss the assumptions behind an SIR model to facilitate the pre-mathematisation step. Moreover, conversations involve both internal and external reflections. On the one hand, students discuss the meaning, appropriateness and function of these assumptions for further mathematisation. They acknowledge that a closed system (fixed N), homogeneity of contacts and cure are simplifications from reality. Still, this compromise results in a point of departure for the eventual enrichment of the model. Further iterations include vaccination, age-group compartments, and splitting of biological-social components.

Repeated experiences with the workshop led to adapt and support the mathematisation of assumptions of the SIR model as seen on task 1, the usual sub-process that presents the greatest difficulty. Hereby, prompting the mathematisation as a system of difference equations bridges the gap from higher-level mathematics (e.g., differential equations), without loss of rigour, since experts do solve and simulate numerically through discretisation. Moreover, the sub-process of mathematical analysis can successfully rely on familiar tools like spreadsheets to facilitate the definition of epidemiological concepts such as a reproduction rate, critical degree of vaccination and herd immunity. Prompts, scaffolding and plenum explanations from the teacher may seem inconsistent with called-for approaches on this paper, such as inquiry-based, problem-solving and landscapes of investigation overall (Skovsmose, 2011). However, these approaches are not equivalent to self-discovery (Hmelo-Silver et al., 2007), and taking the time to crystallise mathematical ideas is a relevant dialogical feature identified by Alrø and Skovsmose (2002).

Some of the students' reflections illustrate broader sociological agendas to rethink our modern risk societies. Here lies the potential of epidemiology to be *critically* exemplary. Students were able to simulate scenarios under different vaccination strategies, showing how it is not the past but a possible threatening future that determines thought and action (Beck, 2000). What is natural and what is a cultural clash, as people's behaviour, change the course of the outbreak. Above all, exploring the models' internal consistencies illustrates its limitations and sensitivities, as we count with limited information to feed the model's parameters as an outbreak unfolds. However, the workshop experience shows that such internal reflections do not coordinate spontaneously with external ones. The issues in Figure 6, in a higher education context, could evoke further discussions in what Kuntze et al. (2017, p. 932) call a "reflection oriented instructional framing". In a way, as with the other perspectives, critical exemplarity is a quality of the example and the teaching.

Mathematical modelling is essential for understanding and handling risk phenomena. Hereby, the COVID-19 crisis – present at the writing of this paper – accentuates the need for a modelling competency in the post-modern globalised societies characterised by Beck as risk societies. In this sense, epidemiology contributes to the justification for mathematical modelling being part of general mathematics education. For the general population, it becomes relevant to know about the dynamics of an epidemic and to have an idea about concepts such as reproduction rate and herd immunity. From a societal point of view, the knowledge base in the population of such matters becomes crucial for political communication about possible strategies for handling healthcare crises and for reaching a necessary level of compliance to recommendations and restrictions. From a democratic point of view, it is essential that there is a sufficiently large part of the population who can act as a critical base concerning authoritative and political communication regarding the risk phenomena at play and related political strategies for handling these risks. The risk society is depending on the educational system to produce a sufficient level of expertise in mathematical modelling for understanding and handling risk phenomena while maintaining democracy.

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