Step by step: simplifying and mathematizing the real world with MathCityMap

Passo a passo: simplificando e matematizando o mundo real com o MathCityMap

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Abstract. Mathematical modelling needs to be taught in realistic contexts. In this paper, we examine the potential of outdoor tasks that are solved by means of the digital tool MathCityMap to learn mathematical modelling stepwise. From a theoretical perspective, it can be assumed that the tasks have the potential to force the "Simplifying/Structuring" and "Mathematizing" steps to occur in an authentic way. In addition, MathCityMap supports the modelling process through hints and an answer validation. In a case study, we compare outdoor tasks with identical indoor tasks to examine in how far these theoretical considerations can be confirmed while using of the system. The results show that the outdoor tasks strongly force the simplifying/structuring step to occur and claim to work as precisely as possible while choosing a model and compensating for inaccuracies. Additionally, the MathCityMap system can support the choice of an exact but still simplified model by means of hints, and triggers an answer validation that might have the potential to make learners rethink the steps of the modelling cycle.
Keywords: modelling; outdoor mathematics; digital tools; MathCityMap; authenticity; validation.

Resumo. A modelação matemática precisa de ser ensinada em contextos reais. Neste artigo, examinamos o potencial das tarefas que são resolvidas em espaços exteriores, usando a ferramenta digital MathCityMap, para aprender o processo de modelação matemática passo a passo. Do ponto de vista teórico, pode supor-se que estas tarefas têm potencial para impulsionar as etapas "Simplificar/Estruturar" e "Matematizar" de uma forma autêntica. Além disso, presume-se que a ferramenta MathCityMap apoia o processo de modelação por meio de pistas e favorece a validação da solução. Num estudo de caso, comparamos tarefas resolvidas em espaços exteriores com as

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mesmas tarefas, mas resolvidas na sala de aula, para averiguar até que ponto essas considerações teóricas podem ser confirmadas na prática com o uso do sistema. Os resultados mostram que as tarefas resolvidas em espaços exteriores provocam fortemente a ocorrência da etapa de Simplificar/Estruturar e aumentam a exigência de trabalhar, com a maior precisão possível, na escolha de um modelo e no controlo de erros. Adicionalmente, o sistema MathCityMap permite apoiar, através de pistas, a escolha de um modelo exato, ainda que simplificado, além de fomentar a validação da resposta, mostrando ter potencial para levar os alunos a rever os passos do ciclo de modelação.
Palavras-chave: modelação; matemática em espaços exteriores; ferramentas digitais; MathCityMap; autenticidade; validação.

Introduction

The real world provides numerous possibilities for mathematical modelling. When mathematical problems are connected to real world situations outside the classroom, students can take a look at an object from different perspectives. To underline the idea of such a problem by using an example, the "Stone in Camps Bay" task is introduced in Figure 1.

Figure 1. The task picture "Stone in Camps Bay"

Let us imagine two different settings for this problem. In the first setting, we present the students with the picture of Figure 1 inside the classroom and ask them to determine the weight of the stone. The picture shows the students the stone from one perspective with the aid of which they can search for a suitable geometric body and obtain the missing data. In this example, this is complicated because the picture does not contain an appropriate object of reference. Still with a direct object of reference, it would be difficult to estimate all the necessary dimensions, i.e. the width of the stone, due to the exclusive perspective of the picture. The alternative situation focuses on the same object, but with the students being on site. Through their presence, the students can move around the object and view it from multiple perspectives. Different choices become possible whereby none of them is suggested by an exclusive perspective in a picture such as in situation 1. Moreover, the data collection on-site is more open in the sense that nearly all the data can be collected – a potential of outdoor modelling tasks.

With the MathCityMap system and its two components, web portal and smartphone app, the idea of outdoor mathematics is realized and digitally enriched. The app guides students along a math trail to real objects and displays a related mathematical problem. Its functionalities are described in more detail hereafter.

Referring to the modelling cycle by Blum and Leiß (2007) in Figure 2, the "Simplifying/Structuring", "Mathematizing" and "Validating" steps are, in particular, relevant for MathCityMap modelling tasks. Simplifying and structuring means to separate important from unimportant information taken from the real situation. Afterwards, this step is followed by the translation of the simplified real situation into a mathematical model. With this mathematical model, students work mathematically and interpret the results. These results are validated with regard to their coherence and appropriateness in relation to the real situation (see Greefrath et al., 2013).

In this article, we examine the potential of outdoor mathematics to strengthen modelling competences from a theoretical and empirical perspective. In a qualitative pilot study with students solving similar tasks indoors and outdoors, we will compare their modelling processes with a special focus on the "Simplifying/Structuring", "Mathematizing" and "Validating" steps. To do so, we categorize the students' solution processes and analyse the products and results. In addition, we examine as to how far the features of the MathCityMap system (i.e., the hints and answer validation) support the individual modelling steps when the students are outside the classroom. Finally, we discuss the preliminary for subsequent research considering the pilot study's limitations.

Mathematical modelling

Mathematical modelling competency

Mathematical modelling competency is defined as

> the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyze or compare given models by investigating the assumptions being made, checking the properties and scope of a given model etc. (Ferri et al., 2007, p. 12)

With this definition and in relation to the modelling cycle according to Blum and Leiß (2007), there are several steps (in this particular cycle, seven) to be completed within the modelling process (see Figure 2).

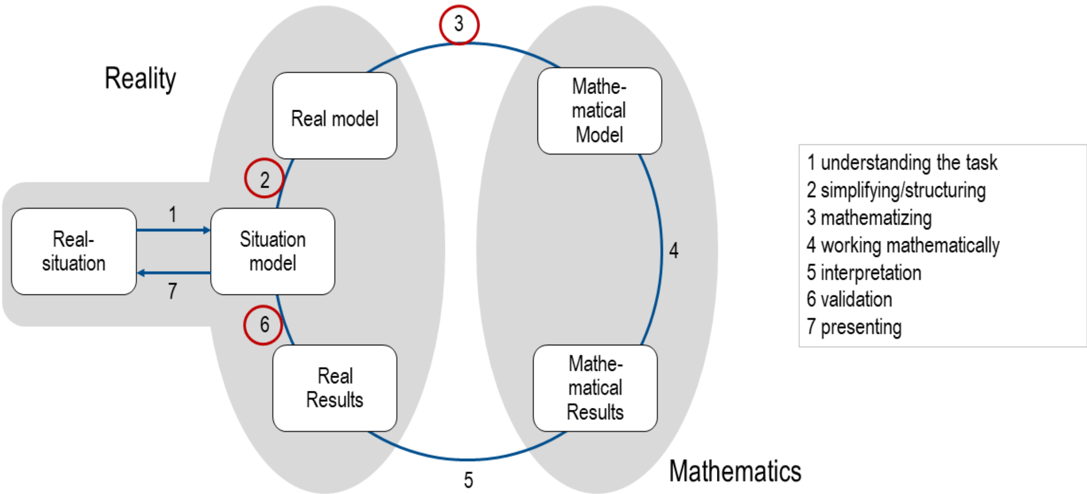


Figure 2. The modelling cycle according to Blum and Leiß (2007)

The modelling cycle idealizes the actual modelling processes. From the research of Borromeo Ferri (2006), it is known that numerous students do not work through the circle one step at a time, but move from the real situation to the mathematical model or change between the real world and mathematics several times. In particular, the "Simplifying/Structuring" and "Mathematizing" steps seem to be a challenge for students within the modelling process (Buchholtz, 2017). In addition, the "Validation" step asks for critical reflections with regard to all the previously considered modelling steps. The result can be validated by a comparison with other investigated results. Especially when the modelling process is not successful, the students might accept that a second try is necessary (Hankeln, 2020). To underline the focus on the three steps, they are highlighted in Figure 2.

The German curriculum for mathematics education refers to the modelling cycle in defining modelling as the change between real situations and mathematical concepts, results or methods and sets up skills concerning each step that students should acquire (KMK, 2015). Being located in Germany, the study and its pilot phase are conducted and interpreted within this context of mathematical modelling.

> The mediation of these central skills is one of the tasks of teachers when dealing with modelling activities, but its instructional implementation can run the risk of being limited to the use of deficient embedded word problems following the strategy: 'Ignore the context, just extract all data from the text and calculate something according to a familiar schema' (Blum, 2015, p. 79). (Buchholtz, 2017, p. 49)

In contrast to the actual idea of mathematical modelling by means of realistic situations, the usage of calculation and schema still seems to be predominant in students' problem solving processes as described. In particular, the mere fact that some modelling tasks do not allow a realistic validation, in the sense of proving the reliability of the result through real data, might support the use of questionable modelling strategies and – in the worst case – the acceptance of arbitrary results.

Mathematical modelling outside the classroom

The mentioned concerns ask for mathematical modelling tasks that prevent students from using primarily a calculation or schema. As Carreira and Baioa (2017) report, a main problem is the "pseudo authenticity" of modelling problems which make arbitrary results acceptable. Following the definition of Vos (2015), a task is defined as "authentic" when its context is a real situation that is not created for the specific school context ("out-of-school context"), and when it has a verifiable task context ("certification"). When students develop mathematical modelling competences, certification in particular seems the best solution for the avoidance of questionable strategies and arbitrary results. According to Vos, the out-of-school context would be the second criterion for authentic tasks. Still, from our point of view, it seems difficult to "teach" students skills related to mathematical modelling completely independently from the school context. Mathematical competences – as defined by the German curricula – are related to mathematical contents, i.e. they can be obtained through mathematical contents. Furthermore, modelling tasks might help in the understanding of mathematical contents and concepts. Therefore, we see a potential in tasks that allow for certification and focus on questions related to situations from the real world with a focus on school mathematics.

On the one hand, it is possible to work on modelling tasks related to a real situation inside the classroom, e.g. by providing a picture and a certification through research on the internet. As Herget and Torres-Skoumal (2007) point out, these tasks have a potential to strengthen thinking and planning skills apart from calculation. Referring to their "Giant Shoe" task, it is necessary to use an object in the picture as an estimator. On the other hand, doing mathematics outside the classroom has a potential for mathematical modelling, i.e. "mathematical ideas, procedures and practices used outside of school may be considered a modelling process rather than the mere set of techniques to manipulate numbers and procedures" (Rosa & Orey, 2017, p. 161).

One possibility for realizing outdoor mathematics is a mathematical city walk (e.g., Buchholtz, 2017), also known as a math trail (e.g., Blane & Clarke, 1984; Ludwig & Jesberg, 2015). The idea behind math trails is a mathematical walk on which interested people are guided to objects and buildings that motivate mathematical questions. The tasks of the math trail can only be solved on site through active mathematical actions, e.g. by measuring or counting (Ludwig et al., 2013).

Barbosa and Vale (2020) highlight the potential of math trails for mathematical modelling: "In a math trail the participants come into contact with realistic problems that highlight the usefulness of mathematics, but more than that amplify the possibility of establishing connections between mathematics and reality" (p. 48). In a study with seventh graders, Buchholtz (2017) identified that by means of math trails "it is possible to create incentives for autonomous mathematising based on real-world problems in a delimited thematic context" (p. 57). Still, he concludes that these tasks are a huge challenge for students, in which we see the necessity of support, e.g. by means of digital tools.

"Digital tools can be of great assistance for teachers and learners, particularly in connection with real-world problems" (Greefrath & Siller, 2017, p. 530), and depending on the used tool and its purpose, different steps in mathematical modelling can be supported by digital tools, for example validating (through feedback on a given answer). Especially in this step, "empirical investigations have shown that independent processes of validation are rarely to be found in students' modelling activities (Blum, 2007). Using digital tools, however, might promote and support these important mathematical activities" (Hankeln, 2020, p. 278). These considerations present two possible benefits of digital tools in the modelling process: digital tools can be implemented directly into the steps of the modelling cycle and digital tools can evoke the implementation of the individual steps in the modelling process. The second benefit, in particular, will be taken into consideration when the MathCityMap digital tool is presented in the context of mathematical modelling.

Mathematical modelling with MathCityMap

Taking up outdoor mathematics on the one hand and exploiting the benefits of digital tools on the other hand, the MathCityMap project leads mobile math trails into an educational context. Its basic idea is the interplay of two components that facilitate the creation and conduction of math trails for teachers and students.

In the MathCityMap web portal (www.mathcitymap.eu; see Figure 3), it is possible to view and create tasks and math trails. The tasks are located closely to the related object through GPS data. A complete task includes a picture of the object, the task itself, hints that the students can call up when solving the problem and a sample solution. For an immediate solution validation, it is possible to choose different answer formats. For measuring and modelling tasks in which small inaccuracies in measuring or a variability in the choice of a suitable model should not lead to a wrong result, the answer format "interval" is an appropriate solution format (Ludwig & Jablonski, 2019).

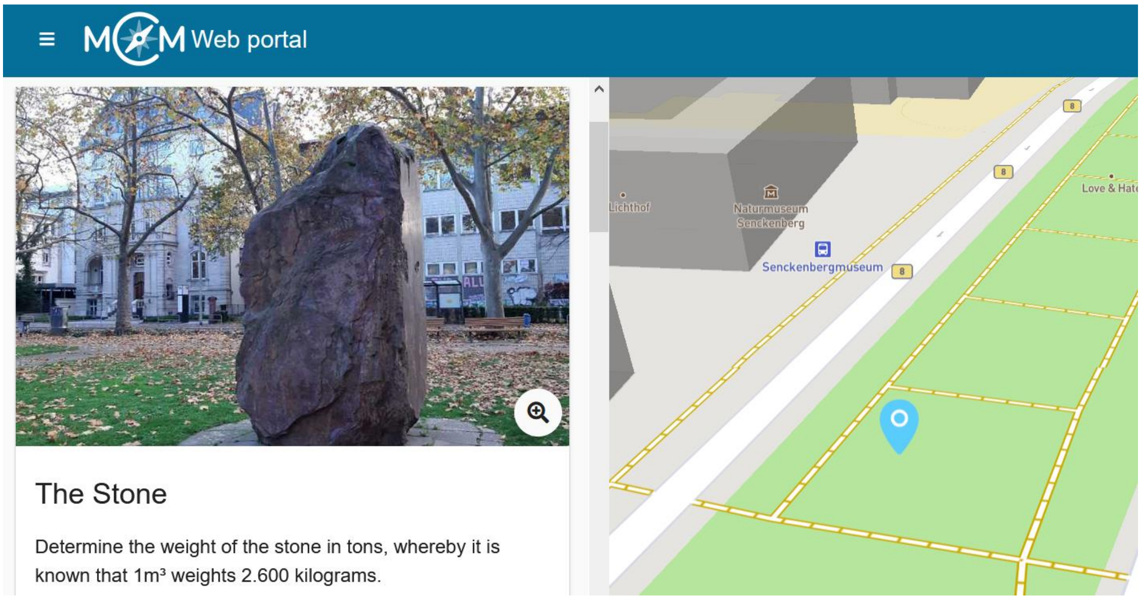


Figure 3. A task in the MathCityMap web portal

The corresponding smartphone app ("MathCityMap" for Android and iOS; see Figure 4) supports students while they walk along a math trail created beforehand in the portal. It shows their own position and the location of the tasks. Furthermore, the tasks previously formulated by the teacher, including the hints, are displayed. Additionally, the app provides a direct feedback with regard to the solution entered and shows a sample solution.

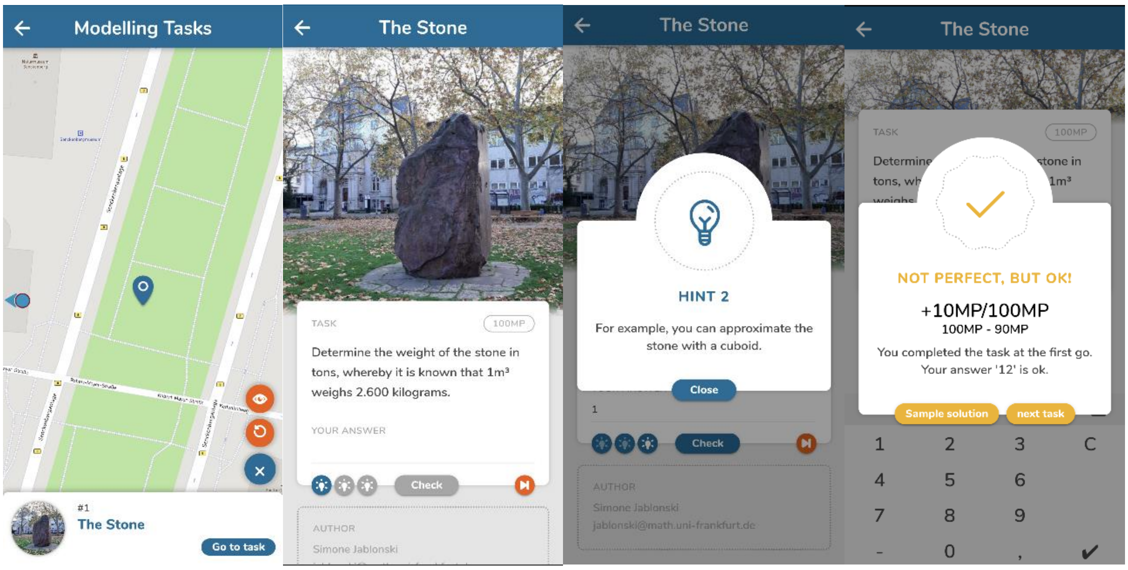


Figure 4. Map, task, hint, and answer validation in the MathCityMap app

Research findings from Germany and Indonesia show that math trail activities with MathCityMap increase the learning outcomes of the students (Zender et al., 2020). With an experimental activity being performed outside and a control group learning inside, the authors see an increased performance in textbook tasks being related to situations that the students came across during the math trail. Still, it remains unclear “what would have happened if we had a control group solve the outdoor tasks on paper inside the classroom” (Zender et al., 2020, p. 11). Hereby, the relevance of a comparison between indoor and outdoor solving processes is strengthened. This will, in the following, be focused on modelling.

For a deeper insight into the outdoor modelling process, we analyse the MathCityMap sample task “The Stone” (see Figures 3 and 4). The task is to determine the weight of a stone in tons, whereby it is known that $1m^3$ of this material weighs 2,600 kilograms. With respect to the “Simplifying/Structuring” modelling step, the task can be solved in different ways. Basically, different mathematical models can be used to solve the problem. Here, the stone could be described by various geometric bodies, whereby differences in the real result can be obtained (see Table 1). On the other hand, it becomes clear that none of these models describes perfectly the shape of the stone which requires simplifications.

Table 1. Results for the task with different models

Real and Mathematical Model	**Real Result**
Cuboid with average values (h=2,75m, l=1,6m, w=1,55m)	17.7t
Cylinder with average values (h=2,75m, r=0,8m)	14.4t
Prism with a trapezoidal area (a=2,45m, c=2,45m, h_t=1,6m, h_p=1,55m)	17.7t
Ellipsoid with average values (a=1,4m, b=0,8m, c=0,8m)	9.8t

The choice of the mathematical model influences the further mathematical procedure. If a cuboid is chosen, the length, height and width must be determined. When approaching through an ellipsoid, the length of the semi axes is required for its volume. It can be assumed that the volume formula of a cuboid will be much more well-known among students than the formula for the ellipsoid volume. Therefore, not only the optimal fit is relevant for the choice of a mathematical model. It also requires further considerations with regard to one's own mathematical knowledge. As soon as the model is chosen, it is necessary to collect the corresponding data. Being equipped with measuring tools, like a tape measure or a folding ruler, the students can take real measurements and do not have to use (solely) estimations as it often happens when solving modelling tasks inside the classroom. Also, in this mathematizing step, the students have to take into consideration what variables can be measured and how inaccuracies might be compensated.

In contrast to the other models, the ellipsoid seems to underestimate the weight immensely. In comparison to the other three models, it does not meet the task's requirement. Nevertheless, the calculation with different models supports the creation of an ideal MathCityMap interval that validates the student's solutions accordingly. During the creation of a MathCityMap task, the author defines an interval for "good" solutions (green interval) and for "acceptable" solutions (orange interval). With respect of the three upper models in Table 1, the green interval for the task, "The Stone", should cover these values. With the estimation of 18 tons (confirmed estimation by Frankfurt's city administration) and several independent measurements and calculations, the green interval is chosen symmetrically from 14–22 tons. By varying the measurements, the results from 12–14 and from 22–24 tons are acceptable, while other results will be marked as "wrong". The app asks the students to rethink the task and check their solution again.

In particular, the aspect that different models and necessary data have to be chosen and collected without any pre-selection can be emphasized outdoors. Through their nature as mathematical tasks using a real object, most of them have a focus on the simplification and mathematization of the real situation in an adequate mathematical model, which corresponds to steps 2 and 3 in the seven-step modelling cycle by Blum and Leiß (2007). Through the hints and task validation by means of an interval, the system might in addition support the validation step. Even though an automatic feedback on the solution quality cannot replace an individual feedback on the solution process, the feedback on a wrong result might create dissonance.

Apart from these theoretical potentials of outdoor mathematics and former research findings, it remains unclear how they can on the one hand be observed and on the other hand be effectively used within and for the modelling processes. Therefore, a more detailed focus is placed on the resulting research questions:

[RQ1] To what extent do the modelling steps "Simplifying and Structuring" and "Mathematizing" differ in an outdoor and indoor modelling process?

[RQ2] In which way does the MathCityMap system support the individual "Simplifying and Structuring", "Mathematizing" and "Validating" modelling steps while solving the task?

Methodology

The formulated research questions are answered in a qualitative case study. In this setting, it is necessary that students solve modelling tasks in a comparison of the indoor and outdoor setting – related to [RQ1]. In addition, the outdoor group uses the MathCityMap app

– related to [RQ2]. For this purpose, a setting for the data collection as well as modelling tasks have been created. To test the suitability of the chosen instruments, a pilot phase took place in December 2020. In a follow-up study in Spring 2022, with a larger sample and school students, the findings will be confirmed or adapted (see Discussion and Summary). In the following sections, the details of the data collection, the analysis of the tasks, and the data evaluation involved will be presented.

The data collection

With the explorative aim to generate hypotheses based up on the formulated research questions, a pilot study with four university students was conducted in December 2020. The students were studying mathematics education in their second year.

The students were divided into two pairs in order to answer [RQ1] on the comparison of the outdoor and indoor setting – one group solved the modelling tasks in the real world with the MathCityMap app and one group solved the same modelling tasks with a reference object for estimations inside the classroom (for a comparison, see Figure 5). The solution processes of both groups were video recorded and transcribed. Furthermore, the students were asked to take notes on their solution processes. In total, the videos last about 115 minutes.

Figure 5. The outdoor task (left) and indoor task (right) "The Stone"

The outdoor group used the MathCityMap app in order to focus on [RQ2]. As described above, the MathCityMap app provides hints so that the students working outside had the option to use them. The students were asked to first solve the tasks without any hints. Only if no progress on the task was made, were they supposed to use the hints given in the following order:

1. The first hint asks the students to find a suitable model, whereby different models are possible and none of them would be perfect. This hint should provoke the "Structuring and Simplifying" step.
2. The second hint gives an example of a suitable model. If this hint is taken, the structuring process is performed by the digital tool.
3. The third hint tells the student to take the missing data from the object. This hint should provoke the "Mathematizing" step.

While using the MathCityMap app outdoors, the modes in which the app was used and the position of the students were tracked with the "Digital Classroom". It recorded the particular event (reading the task, taking hints, entering solutions) together with the corresponding location and time.

Detailed analysis of the task, "The Stone"

The study includes seven modelling tasks with objects located close to the Bockenheim Campus of Goethe University Frankfurt. Their analysis in terms of their difficulty happens by means of the thought structure, i.e., the individual thinking steps that are necessary to solve the task and the links between the steps (Reit, 2017).

In Figure 6, the solution process for the outdoor task, "The Stone", is presented under the assumption that a cuboid is chosen as mathematical model. The blue parts present the necessary measurements, yellow parts symbolize calculations, and grey parts contain information that can be taken from the task formulation (Reit, 2017).

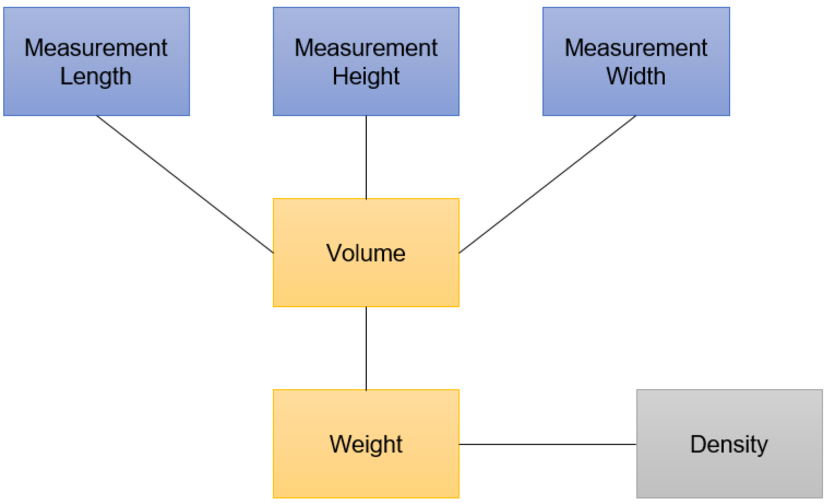


Figure 6. Solving process for the outdoor task "The Stone"

In comparison to the outdoor situation, instead of measuring, the students inside the classroom have to estimate the length, width, and depth in reference to the person standing in front of the stone. The structuring and mathematizing (by means of a cuboid) involves two additional steps, i.e., the estimation of the person's height and the proportion that results from this estimation (see Figure 7).

In the Appendix 1, an overview of the remaining six modelling tasks that were used in the pilot study is given. From their nature, they are all embedded in a geometric context. Still, they differ in the choices of the possible models and mathematizations which are highlighted. As for the exemplary task, "The Stone", they are all created in two ways – as an outdoor task in MathCityMap and as an indoor task with an object of reference. Especially for the indoor setting, the picture is highly relevant for solving the task. With reference to Herget and Torres-Skoumal (2007), the person standing in front of the actual task object can be used as an estimator (see Figure 5, right). The attached table shows the pictures of the outdoor setting without any object of reference.

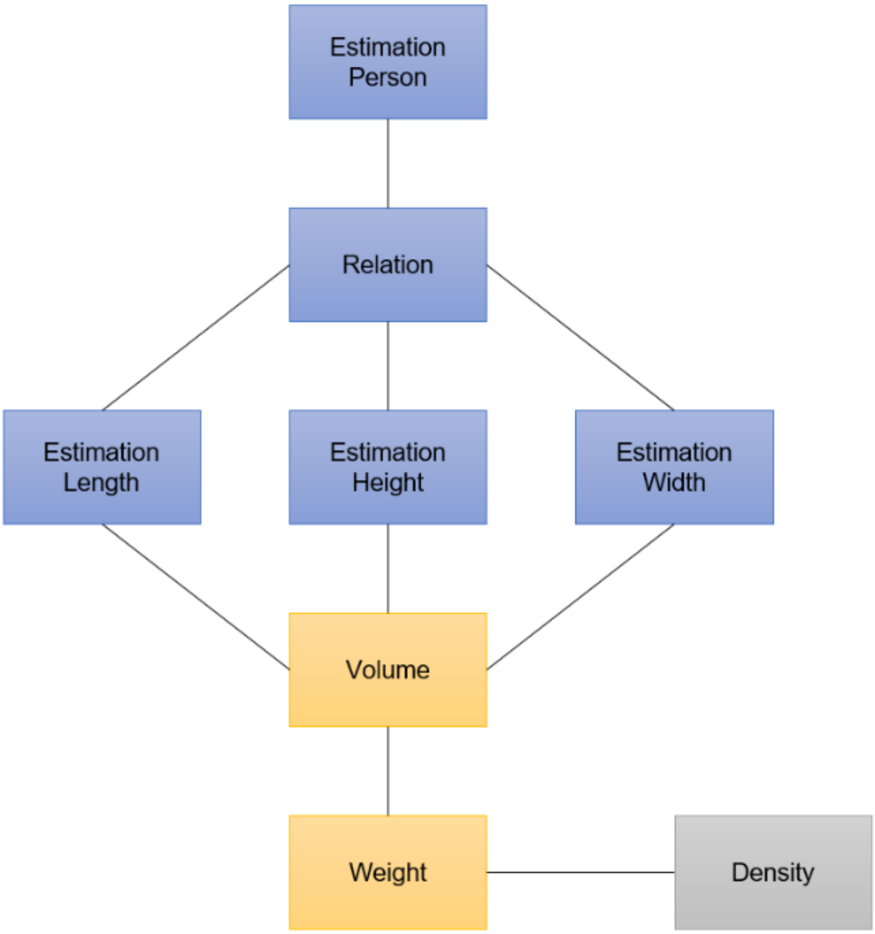


Figure 7. Solving process for the indoor task "The Stone"

Data analysis

The conversations, the written solutions, and the MathCityMap events in the Digital Classroom were taken into consideration for the data analysis. To analyse the *solution processes*, the conversations are analyzed by means of a qualitative content analysis. This inductive analysis results in the following categories and examples taken from an indoor task “The Stone” (see Table 2).

The “Simplifying/Structuring” step is hereby divided into the choice of a real model, the compensation for inaccuracies and making assumptions. These activities facilitate the real situation and provide a basis for the “Mathematizing” step. In this step, the focus is on the creation of a mathematical model, i.e., the choice and collection of the necessary data. In addition to these activities, the students tend to show considerations in the other modelling steps as well, especially in “Understanding” and “Validating”. These steps are defined in the modelling cycle by Blum and Leiß (2007).

Table 2. Modelling activities for the “Simplifying/Structuring” and “Mathematizing” steps

Modelling Step	Modelling Activity	Example
Simplifying/Structuring	Choose a Real Model	“It seems to be a cuboid with a cut off edge”.
Simplifying/Structuring	Compensate inaccuracies	“We could take the mean of the small and the big height”.
Simplifying/Structuring	Make assumptions	“The person might be about 1.65m”.
Mathematizing	Choose the necessary data for the Mathematical Model	“For the cuboid’s volume, we need its height, length and width”.
Mathematizing	Collect the necessary data for the Mathematical Model	“So the big height would be twice 1.65m”.

For their *written solutions*, a scale consisting of different levels in accordance with the modelling cycle by Blum and Leiß (2007) is used (Ludwig & Reit, 2013). Six levels are distinguished whereby each level involves an additional step of the modelling cycle in comparison to the previous level (Figure 8):

Level 0
- students do not understand the performance requirements or refuse to work on the task

Level 1
- students understand the given real situation
- students are not able to structure or simplify it

Level 2
- students develop a real model
- students are not able to transfer this model into a mathematical model

Level 3
- students transfer the real model into a mathematical model
- students are not able to complete the solution process

Level 4
- students are able to work in the mathematical context
- students do not validate the result

Level 5
- students validate the result and give suggestions for improvement

Figure 8. Scale for written solutions

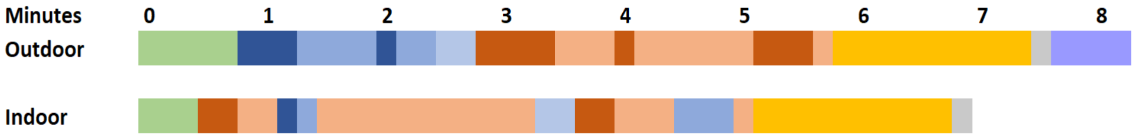
In addition, the *MathCityMap events* from the outdoor group are taken from the Digital Classroom of MathCityMap. The quantity of the hints used and sample solutions provides insight into the usage of the app and the role of the digital tool in the modelling processes outdoors.

Results

As identified in the study's methodology and data analysis, the results of the study will be presented on three different levels. Firstly, we focus on the students' solving processes by means of the video and audio material (duration and qualitative contents). Secondly, the students' notes are taken into consideration for the analysis of the written results, i.e., in terms of the explicitness of the modelling steps and the solution quality. Finally, for the outdoor group, the app events are presented.

Duration of the modelling processes

By analyzing the solving processes of the students, we take into consideration the categories of the identified modelling activities. All five categories identified in Table 2, "Choose a Model", "Compensate Inaccuracies", "Make Assumptions", "Choose the necessary Data", and "Collect necessary Data for the Mathematical Model", are relevant for both indoor and outdoor modelling tasks. With Figure 9, we firstly focus on the intensity of these steps (see also Ärlebäck & Albarracín, 2019). It gives an overview of the identified modelling steps in the task, "The Stone". The activities in "Simplifying/Structuring" and "Mathematizing" are divided into the categories from Table 2. Still, the choice of colour (deep and light blue, and red) should symbolize their relationship.



	Outdoor	Indoor
1 Understanding	10%	7%
2.1 Simplifying/Structuring: Choose Model	8%	2%
2.2 Simplifying/Structuring: Compensate Inaccurancies	12%	10%
2.3 Simplifyling/Structuring: Make Assumptions	4%	5%
3.1 Mathematizing: Choose Data	16%	9%
3.2 Mathematizing: Collect Data	20%	40%
4 Working Mathematically	20%	24%
5 Interpreting	2%	2%
6 Validating	8%	0%

Figure 9. Comparison of the modelling processes outdoors and indoors for "The Stone"

From the durations, it can be observed that especially the categories in the "Simplifying/Structuring" modelling step differ in the outdoor and indoor setting. While solving the same task, the outdoor group spends nearly two minutes discussing a suitable real model, the compensation of inaccuracies and making assumptions (deep and light blue, in total about 24% of the whole modelling process). The categories are relevant indoors as well, but not to the same extent. Inside the classroom, the students talk for about 30 seconds about the model, inaccuracies, and assumptions which is about 17% of the total modelling process. In particular, the choice of a model takes less time inside the classroom. For "Mathematizing", the students inside the classroom start collecting the data early. They take about four minutes for this step in contrast to three minutes outdoors. In particular, the process of collecting data takes more time indoors, namely 40% of the total modelling process. The other modelling processes are nearly similar in the inside and outside setting despite the validation step which is only relevant in the outdoor group. This issue is taken up later in the analysis of the app events in MathCityMap. The mean values of the durations measured for the remaining tasks confirm the results presented here as an example, despite "Simplifying/Structuring: Make assumptions" which seems to be more relevant in the indoor setting. See Appendix 2 for a detailed analysis by means of diagrams and mean values for all tasks.

Qualitative comparison of the modelling processes indoors and outdoors

In the following, we focus on a comparison of the outdoor and indoor modelling processes on a qualitative level. The theoretical considerations as well as the differences in duration lead to the assumption that the steps taken outside differ from the steps taken inside the classroom. For the “Simplifying/Structuring” step, this assumption can be confirmed for the following aspects:

a) Analysis of the situation from a holistic point of view

The students solving the task outdoors have more possibilities to choose a suitable model, i.e., through walking around the object. For example, the students solving “The Stone” outside start their process of choosing a model by “let us firstly walk around the stone”.

b) Forward planning on the basis of mathematical knowledge and data collection

From the example of the students discussing their model for the outdoor task, “The Base” (see Appendix 1), an interesting approach can be observed.

Student 1:	We have a cone that is cut off. Do you know the formula for the truncated cone?
Student 2:	No, I do not know it. [...]
Student 1:	Then we take a large cone and cut off a small one.
Student 2:	The only question is how we get the point [the top].

While choosing the model, the students already take into consideration their mathematical knowledge, i.e., they do not immediately know the formula for a truncated cone. In addition, they consider which data they would have to measure and if this would be possible.

c) Work precisely and effectively at the same time

The possibilities of involving different perspectives and collecting nearly all data seem to motivate the students to search for a very precise model, i.e., “We would need much longer for counting, but it would be more precise” when solving the task “Stones” (see Appendix 1). Still, it is their aim to work effectively at the same time. A discussion on the balance of precision and effectiveness in the choice of a model and the compensation of inaccuracies can be observed in the solution of the task “The Stone”:

Student 1:	It is a cuboid with a cut-off edge. [...] I think that we can compensate for it by dividing the stone into two cuboids. A big one and a small one above. [...] So we would need two different heights I think.
Student 2:	The question is [...] Do you think that it will make such a big difference if we simply measure one height?
Student 1:	Do you mean as height for the whole stone?
Student 2:	Yes, I do not think that it changes the result immensely.
Student 1:	Okay, so the result does not have to be perfect, right?

These three identified aspects concerning the “Simplifying/Structuring” step are the relevant differences that the outdoor group shows in comparison to the indoor group. The analysis does not only confirm the theoretical considerations, but underlines the longer duration of this modelling step outdoors. Interestingly, the different processes do not primarily influence the choice of a basic model, i.e. both student groups choose similar models, e.g. a cuboid for the stone, but the following modelling steps.

When it comes to "Mathematizing", the students outside the classroom take measurements on the object, while the students inside the classroom estimate the lengths by means of the referring object, e.g. comparing by showing. Both activities include discussions, i.e. which data are needed and how the measurement takes place or how to make an estimate. Nevertheless, the students inside the classroom discuss their estimations more intensely and this is reflected in the longer duration, especially in the "Collecting Data". With their knowledge that the objects exist in reality, the students inside also show the willingness to be as precise as possible but know at the same time about their limitation of "not being at the object's location".

From the processes in the "Simplifying/Structuring" and "Mathematizing" steps, the following can be concluded: Both task formats force the previously named steps to occur in the modelling process. The outdoor tasks have a focus on "Structuring", i.e. in forcing mathematical discussions on a model, while the indoor tasks primarily force discussions in the subsequent "Mathematizing" step on their estimations. The "Validating" step is considered in the data analysis of the MathCityMap app events.

Analysis of the written solutions

By means of the different levels defined by Ludwig and Reit (2013; see Figure 8) and (only) the written solutions, it can firstly be concluded that the students – indoors and outdoors – reach Level 4 for all tasks. They are able to understand the given real situation, develop a real model, transfer it to a mathematical model and come to a result. A written validation is neither visible in the indoor nor in the outdoor group's solutions. For the sample task, "The Stone", the written solutions indoors (above) and outdoors (below) are presented in Figure 10.

Apart from both groups reaching Level 4, the comparison shows that the group being outdoors takes fewer notes on their solution process. In contrast, the group indoors even integrates a drawing that shows the chosen model and explains their process of obtaining the data by estimation and relationship. The fact that the students outside the classroom do not take as many notes as in the inside setting can be confirmed by a general analysis of the math trail guides with MathCityMap tasks. For the students, the setting outside the classroom seems to differ from their usual tasks in mathematics teaching and side-line the presentation of the task solution process. Furthermore, the solution checks and storage in the app replaces the necessity to note down the result.

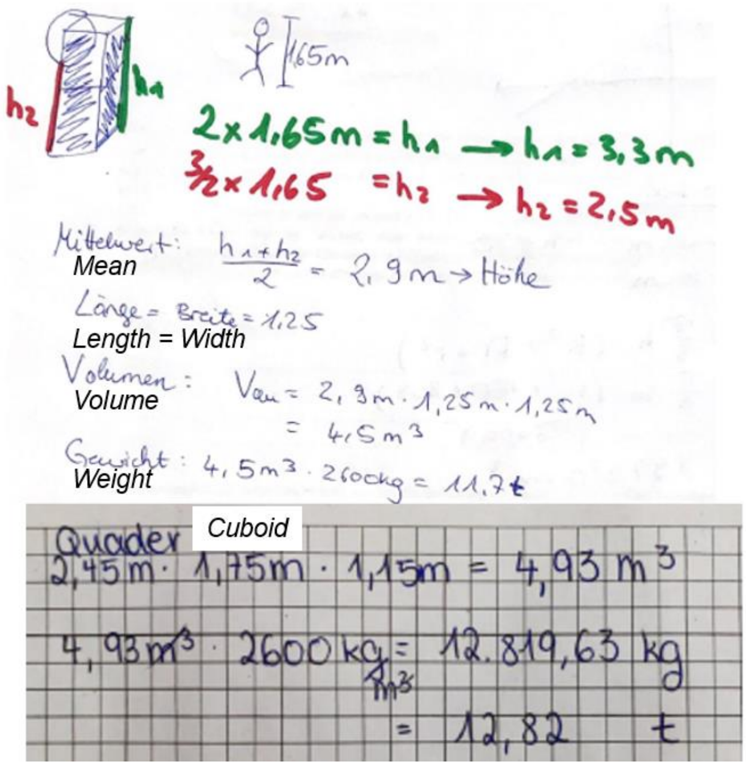


Figure 10. Written solutions for the task "The Stone" (above: indoor; below: outdoor)

From the written solutions, it is further possible to compare the solution qualities of both groups. The solution for "The Stone" by the indoor group is 11.7 tons whereby the outdoor group produces a solution of 12.8 tons. Taking the interval defined beforehand in the MathCityMap app as a reference, the indoor result is "wrong", whereby the outdoor result is acceptable, both being lower than the interval chosen between 14 and 22 tons. With the possibility of taking measurements at the real object, the students outdoors achieve a good solution in all their further tasks. The students inside have to estimate the corresponding data which leads to two results not being perfect but rather ok and "false" results. Of course, the result validation has to be seen in the context of the defined MathCityMap interval.

As presented in Table 1, the intervals are based on real measurements which are not accessible in the indoor modelling setting. Having only one perspective and the object of reference, the results indoors cannot be as precise as in the outdoor context. For example, in the task, "The Stone", the width can hardly be estimated by the picture. In addition, the students did not validate their result with another model. The solution quality is therefore solely an indicator of the precision and the comparison has to be interpreted within the context of this limitation.

The role of the MathCityMap app in the outdoor setting

Finally, we analyse the usage of the MathCityMap app in the outdoor modelling setting through the events being tracked in the Digital Classroom. Apart from the navigation and the object identification, we can identify the following features as relevant for the modelling process.

The hints that MathCityMap provides were called up only in one case. This situation appeared while the students were searching for a precise and effective model for the task "Stones". Before asking for a hint, they decided to count the number of more than 1,000 stones – which would have been a questionable strategy. After reading the hints, the students decided to model the area by a semi-circle and count the stones in the radius. With "stones" as unit in the formula of a semi-circle, it was easy for them to calculate the total number of stones. In this case, the hints supported the choice of a model in both preciseness and effectiveness.

Further, the automatic result validation in the MathCityMap app triggered the students to immediately analyse the quality of their results. Even though it did not happen in the sample of this study due to acceptable or good results, it can still be assumed that a negative feedback would lead to a rethinking of the steps in the modelling cycle. This assumption is made from the case in which the students receive a feedback in the "not perfect, but ok" interval for the task "The Stone". The students took a moment rethinking their modelling process and compared their solution with the sample solution.

Student 1: Yes, it [the result] is okay. Maybe a little bit too heavy?
Student 2: Let's see the sample solution.
Student 1: 245 is the same height as ours.
Student 2: Yes, but they take the mean for the height. So we have maybe underestimated the height a little bit too much.

Before checking the sample solution, the students assumed that their result was too heavy. After a comparison, they noticed the difference in height. Whereby the students worked with the smallest height, the sample solution contained the mean of the smallest and biggest height. Still, they receive an acceptable solution because of an appropriate choice of a mathematical model. The sample solution helps the students to validate their result and in the end, they realize that their result underestimated the weight of the stone.

An additional observation underlines the importance of the sample solution. In all outdoor modelling tasks, the students had a look into the sample solution – even after receiving a positive feedback. Coming back to the idea of a realistic validation of results, this is an important indicator for more precise results in outdoor modelling tasks. In a repeated use, this might lead to a more elaborated sense of scale for estimations.

Discussion and summary

In this paper, we have focused on the potential of outdoor modelling tasks with the digital tool MathCityMap. In this study, we have chosen to focus first of all on differences in the indoor and outdoor modelling setting [RQ1], and secondly on the role of MathCityMap in the outdoor modelling process [RQ2]. Through the nature of outdoor mathematics and the functionalities of the MathCityMap app, the modelling process was mainly limited to the "Structuring and Simplifying", "Mathematizing" and "Validating" steps. From the comparison of the indoor and outdoor setting, the following preliminary findings on [RQ1] can be stated:

1. Both – outdoor tasks with a real object and indoor tasks with a picture including a reference object – involve the "Structuring and Simplifying" and "Mathematizing" modelling steps which can be related to the research findings by Buchholtz (2017).
2. The basic chosen models are often similar in the indoor and outdoor context.
3. The way in which the students structure the situation, i.e. which model should be chosen, is discussed more intensely by the students being outdoors. This is characterized by taking different perspectives and by discussions in which data and knowledge are needed and in which data can be gathered. Through the possibility of measuring the real object, the students try to be as precise as possible and do not take any estimations. Still, it is their aim to work effectively.
4. The students inside the classroom have fewer possibilities to choose a model because of the limited data that can be gathered and by having only one unchangeable perspective of the object. In contrast to shorter discussions on the situation model, the indoor setting leads them into more intense discussions on the assumptions and estimations that they have to make in order to gather the necessary data for the mathematical model. This finding confirms the theoretical differences of the outdoor and indoor tasks by means of the additional thought structures according to Reit (2017).
5. Both task formats enable the students to proceed successfully through the "Structuring/Simplifying" and "Mathematizing" steps, i.e. through the transfer of the situation into a mathematical model. In all cases, they produced a result. The results are only validated in the outdoor setting, as a result of which we assume that what needs to be discussed in [RQ2] is prompted by the MathCityMap app.

The following hypotheses concerning [RQ2] and the potential of the features in the MathCityMap app can be formulated:

6. The hints in the MathCityMap app can be helpful if the students are overstrained by the possibilities that their environment provides. The hints formulated in this study support a balance between the students' demand on preciseness on the one hand and an effective solving process on the other hand.

7. The solutions outside are more precise, i.e. through the potentials discussed in [RQ1]. The possibility to check the result in the MathCityMap app helps the students in performing the validation step. In addition, the sample solution seems to be an important feature of the app and the result validation. We assume that many students would repeat the steps previously carried out if the result were not appropriate. The automatic answer validation could therefore be an advantageous element in the modelling process, also in contrast to the same tasks being presented indoors. This aspect can be related by the research findings of Zender et al. (2020) concerning differences in the indoor and outdoor setting with similar tasks. Still, we cannot provide any findings on the differences in the performance of the students.

To conclude, the results show the potential of involving especially the structuring process of mathematical modelling tasks outdoors. Having the theoretical potentials and empirical findings in mind, the real world forces the students to make appropriate simplifications, choose from all the possible data, and make the actual measurements.

This challenging task can be supported thanks to a digital tool and its possibility to give hints and validate the solution. A benefit seems to be the willingness of the students to work precisely and accurately, even though it is clear that the real world cannot be perfectly described by a mathematical model. As a result of discussions on the basis of their mathematical knowledge and hints, the students can find a balance and achieve realistic results. In the next step, these results will provide a meaningful basis for validating the results which is fostered by the answer validation.

Due to the context of the pilot study, the results presented need to be confirmed. The small sample does not allow any representative conclusions or generalizations. Still, the findings give some first ideas on the differences in outdoor and indoor modelling on the one hand, and the role of the digital tool MathCityMap on the other hand. In the Spring of 2022, the main study with the title “MAP – Modelling, Arguing and Problem Solving in Outdoor Mathematics” will take up these results and conduct the math trail with school students and a bigger sample in order to confirm or adjust the findings.

As a main finding from this pilot phase, we conclude that MathCityMap outdoor modelling tasks possess the potential for students to gain modelling competences stepwise – with a focus on “Simplifying/Structuring”, “Mathematizing” and “Validating”. Moreover, this happens in a real life context and with the support of a digital tool. In this combination, outdoor modelling tasks can be a promising enrichment for the development of modelling competences in schools and universities.

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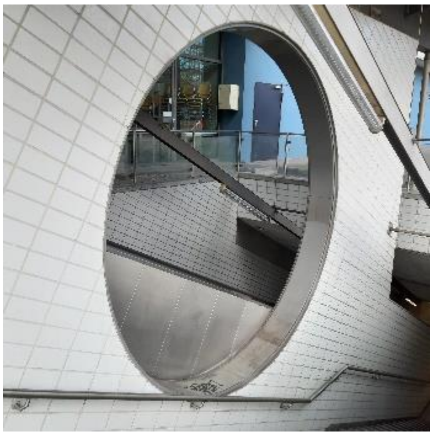
Appendix 1: The MathCityMap Tasks

Task Formulation and Picture	Possible Models and Mathematizations
Task “Stones”: Determine the number of stones inside the grey boarder. 	The area can be modelled by means of a semi-circle. Option 1 (Idea to define a new length unit): Determine the number of stones for the radius and insert it into the formula for the area of a semi-circle. Option 2 (Idea of Proportions): Calculate the area of the semi-circle and determine the number of stones per square meter.
Task “The Dino”: Determine the shoe size of the dino. 	The shoe size can be determined by means of the own shoe size and foot length. Option 1 (Idea of Proportions): Measure the own foot length and determine a ratio with the shoe size. Option 2 (Idea of Comparing): Determine how often the own shoe would fit into the length of the dino’s foot.
Task “Age of the Tree”: Determine the age of the plane tree. It is known that an 52 year-old plane tree has a diameter of 40 cm. 	The tree can be modelled by means of a cylinder. Option 1 (Idea of Circumference): Measure the circumference of the tree and calculate the diameter which gives the age using proportions. Option 2 (Idea of Diameter): Approximate the diameter and use proportions to determine the tree’s age.

Task "The Hole": Determine the circumference of the hole. Give the result in meters.



The hole can be modelled by means of a circle.

Option 1 (Idea of direct measure): Measure the diameter and calculate the circumference.

Option 2 (Idea of Counting): Determine the diameter by measuring one brick's length and counting the bricks needed for the diameter.

Task "The Underground": Determine the slope in which the underground comes out of the ground. Give the result in percentage.

Different options to determine vertical and horizontal differences.

Option 1 (Idea of Gradient Triangle): The slope can be modelled by means of a gradient triangle.

Option 2 (Idea of Similar Triangles): Measure the legth of the roof and the length of one pillar.

Task "The Base": Determine the volume of the stone base. Give the result in m^3.

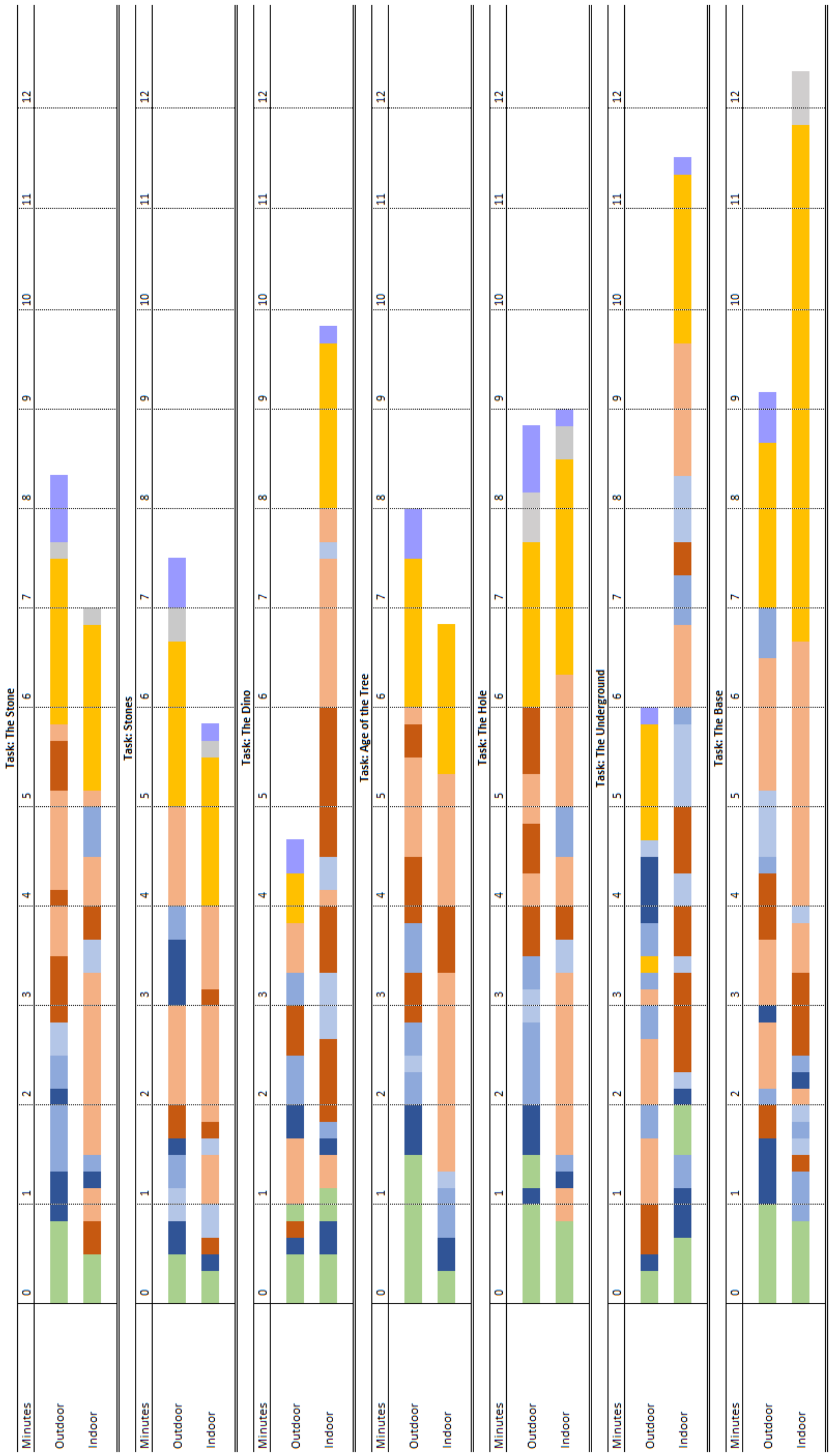
The base can be modelled by means of a truncated cone.

Option 1 (Idea of Truncated Cone): Use the volume formula of a truncated cone.

Option 2 (Idea of two Cones): Use the difference of a big and a small cone.

Option 3 (Idea of two Cylinders): Use the mean of a big and a small cylinder.

Appendix 2: Modelling Activity for all Tasks



	Outdoor (mean values)	Indoor (mean values)
1 Understanding	11%	7%
2.1 Simplifying/Structuring: Choose Model	10%	3%
2.2 Simplifying/Structuring: Compensate Inaccurancies	13%	6%
2.3 Simplifyling/Structuring: Make Assumptions	4%	8%
3.1 Mathematizing: Choose Data	13%	13%
3.2 Mathematizing: Collect Data	21%	36%
4 Working Mathematically	19%	23%
5 Interpreting	2%	2%
6 Validating	7%	1%