

Mathematical modelling in scientific contexts and in Danish upper secondary education: are there any relations?

A modelação matemática em contextos científicos e no ensino secundário dinamarquês: existem relações?

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Abstract. Mathematical modelling and applications have been agreed upon as a justification for the teaching of mathematics at several levels of educational systems across the world – and as a core element of the teaching itself. Despite several theoretical constructs describing the teaching and learning of mathematics, we still face challenges regarding its learning, what should be learned, and if it should reflect scientific practices. Didactic transposition theory offers an approach for how to analyse the relation between the scholarly knowledge, notions and practices that motivated the knowledge taught at school and what shaped it. In this paper we analyse the external didactic transposition through modelling cases from the 20th century and the framing of modelling in Danish upper secondary mathematics. We use our analysis to discuss if potentials are lost in the transposition and how these can be brought into play.

Keywords: didactic transposition; history of mathematics; mathematical modelling; upper secondary mathematics education.

Resumo. A modelação matemática e as aplicações têm merecido concordância enquanto argumentos para o ensino da matemática, em vários níveis, nos sistemas educacionais em todo o mundo, sendo igualmente considerados como um elemento central do próprio ensino. Apesar da existência de vários construtos teóricos que descrevem o ensino e a aprendizagem da matemática, continuamos a enfrentar desafios em relação à sua aprendizagem, ao que deve ser aprendido e se isso deverá refletir as práticas científicas. A teoria da transposição didática oferece uma abordagem para analisar a relação entre os saberes académicos e as noções e práticas que deram origem aos saberes ensinados na escola e em que moldes. Neste artigo, analisamos a transposição didática externa por meio de

situações de modelação do século XX e o enquadramento da modelação matemática no ensino secundário dinamarquês. Usamos a análise produzida para discutir se as potencialidades são perdidas na transposição e perceber como podem ser efetivadas.

Palavras-chave: transposição didática; história da matemática; modelação matemática; educação matemática no ensino secundário.

Introduction

The notion of mathematical model and modelling took form and entered the discourse of mathematics during the 20th century with the increasing mathematization of a variety of fields such as economics, climate science, biology, medicine and more. The emergence of mathematical modelling is now reflected in mathematics curricula across the world, where it takes the role of content knowledge and is part of the justification for the teaching of mathematics (Blum et al., 2007). In educational research various approaches, theoretical developments and empirical studies on how to teach and learn mathematical modelling have been developed (e.g., Ärlebäck & Doerr, 2015; Barquero et al., 2013; Blomhøj & Kjeldsen, 2006; Niss & Blum, 2020). One theoretical construct is the modelling cycles (e.g., Niss & Blum, 2020), which depict sub-competencies and processes involved in modelling activities. The cycles have taken prominent roles in curricula documents, as we show below. Still, it has been questioned if they depict the reality of modelling (Biehler et al., 2015). Educational researchers have argued that “different variations of the modelling cycle are analytical tools for analysing the sub-processes . . . involved in mathematical modelling competency” (Blomhøj, 2011, p. 343), the cycles are not meant to depict reality. Without taking sides, we believe this raises a central question concerning what relation, if any, can be found between mathematical modelling inside and outside of the school context.

To pursue this question, we draw on the notion of didactic transposition, which stems from the work of Chevallard (1985). The didactic transposition is characterised as the “transformations an object or a body of knowledge undergoes from the moment it is produced, put into use, selected, and designed to be taught until it is actually taught in a given educational institution” (Chevallard & Bosch, 2020, p. 214). The transposition is often depicted as in Figure 1: *scholarly knowledge* represented by research institutions and other producing institutions; *knowledge to be taught* shaped by the *noosphere* and accessible through curricula documents; the *taught knowledge* that can be observed in the classroom and teachers’ lesson plans, and which also provide insights about *learnt knowledge*, which further can be studied in the work done by the students. The study of those components for a specific topic in a specific educational context provides an epistemological reference model that allows us to question choices made in the educational system. The transposition between scholarly knowledge and knowledge to be taught is the external transposition,

which we wish to analyse for mathematical modelling. This kind of analysis enlarges the empirical unit of analysis when we pay attention to the nature, the origin and the initial function of the mathematics that is being taught and learned in school (Bosch, 2015). Thus, we study the external didactic transposition of modelling from when it emerged in research to its place in school today.

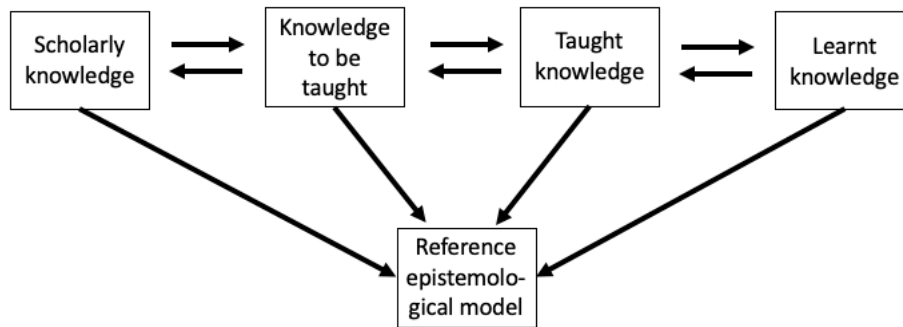


Figure 1. The didactic transposition of knowledge in school systems (according to Bosch and Gascón (2014, p. 70))

A first step towards an external didactic transposition of mathematical modelling has been taken by Frejd and Bergsten (2016). They investigated modelling as a professional task in the workplace representing scholarly knowledge as it is produced and used in private companies and research. They argue that modelling as a professional task is highly compartmentalised and identified major differences between modelling in workplaces and in school. They concluded that the influence of scholarly knowledge on school mathematics in Sweden is weak. In the present paper we conduct a similar investigation but with focus on modelling as a scientific practice. Our approach aligns with Wijayanti and Bosch (2018) by drawing on historic documents. Our analysis of such documents is not complete but captures traits of scholarly knowledge regarding the development and uses of mathematical modelling in scientific contexts. Our study has been guided by the following question: Based on the external didactic transposition, what relation – if any – can be identified between mathematical modelling in scientific contexts, as represented by three different cases from the 20th century, and the framing of teaching mathematical modelling at upper secondary level in Denmark?

Our analysis is divided into two parts: first we present the analysis of one historic case from the mathematization of economics and two cases from biology, and second, we analyse the knowledge to be taught in terms of ministerial guidelines, textbooks and exam exercises from Danish upper secondary school. The analysis is guided by the didactic transposition methodology and variables we present below.

Methodology

As Wijiyanti and Bosch (2018), we adopt the didactic transposition methodology as described by Bosch and Gascón (2014). To analyse practices through historic examples is well known in other research communities, where such studies shed light on practices applying mathematics and the entanglements of mathematics with methods of other research fields (see, for example, Dalmedico, 2001; Knuuttila & Loettgers, 2017). Our cases represent early attempts of the migration of mathematics into new fields of inquiry in which mathematical modelling currently plays a significant role as a scientific practice. The model constructions are accessible for non-specialists and the cases provide a window into “modelling in the making”. The modelling was subject to criticism from researchers in the target discipline, which gives us insights into barriers of understanding, differences in approaches to knowledge production, and discussions of epistemic value of modelling across disciplines.

Bosch and Gascón describe knowledge to be taught as that which “can be accessed through official programs, textbooks, recommendations to teachers, didactic materials, etc.” (2014, p. 71). In Denmark, the teaching of mathematics is regulated by the national curriculum including the high stake exit exam. In this paper we restrict ourselves to general upper secondary school (gymnasium), which is attended by approximately 60% of the youth (Danish Ministry of Education, 2021).

We have constructed our model of analysis (Table 1) by combining Bosch’s (2015) emphasis on the need to pay attention to the motivation, origin, and function of what is to be taught and learnt in the didactical transposition with elements from Boumans’ (2005) methodology of models¹. Boumans argues that there are implicit criteria (satisfying mathematical or other requirements, being useful for policy, etc.) that models have to meet. This is done by integrating various *items* (like baking a cake with various ingredients) in the model construction. The items include analogies, mathematical concepts and theories, theoretical notions, and empirical data, which are the ones we have used in our analyses (see Table 1). Regarding the model function, we use Gelfert’s (2018) notion of *exploratory modelling*, which will be explained below. Furthermore, we have added *modelling strategy* and *discussion of epistemic value* in our model of analysis as important aspects for understanding disputes between various actors.

Table 1. The variables and categories guiding our analysis of scholarly knowledge and knowledge to be taught

Meta aspects		Items used in modelling construction				Model function	
Motivation	Modelling strategy	Discussion of epistemic value	Analogies	Mathematical concepts and theories	Theoretical notions from other areas	Empirical data	Explorative function (1-3)

Analysis I: scholarly knowledge of mathematical modelling activities

Economics and new mathematics

Our first example is von Neumann's (1937) paper *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*. Historians of economics have coined this paper "the single most important article in mathematical economics" (Weintraub, 1983, p. 13). It was translated into English in 1945 with a change in title: "A Model of General Economic Equilibrium". Von Neumann considered a general economy where n goods (G_1, \dots, G_n) are produced by m processes (P_1, \dots, P_m), and he asked the question: "Which processes will be used (as 'profitable') and what prices of the goods will be obtained?" He mathematized the problem as a system of six linear inequalities:

$$x_i \geq 0 \quad (1)$$

$$y_j \geq 0 \quad (2)$$

$$\sum_{i=1}^m x_i > 0 \quad (3)$$

$$\sum_{j=1}^n y_j > 0 \quad (4)$$

$$\alpha \sum_{i=1}^m a_{ij} x_i \leq \sum_{i=1}^m b_{ij} x_i, \text{ for all } j \quad (5)$$

where $y_j = 0$ if ' $<$ ' holds.

$$\beta \sum_{j=1}^n a_{ij} y_j \geq \sum_{j=1}^n b_{ij} y_j, \text{ for all } i \quad (6)$$

where $x_i = 0$ if ' $>$ ' holds.

Here a_{ij} (expressed in some unit) denotes the amount of G_j that is consumed in the process P_i , and b_{ij} denotes the quantity of G_j that is produced by the process P_i . The intensities of the processes are (x_1, \dots, x_m) while (y_1, \dots, y_n) represent the prices of the goods. Finally, α is the expansion factor and β is the interest factor. The inequalities in (5) assure that it is impossible to consume more of the good G_j than the amount produced. If less is consumed, G_j becomes a “free” good and its price $y_j = 0$. The inequalities in (6) indicate that there is no profit in the model – everything gets re-invested. If there is a loss, i.e., if strict “>” holds, the process P_i will not be used and $x_i = 0$ (von Neumann 1937, pp. 75-76). This mathematization turned the question of existence of economic equilibrium into the question of existence of a solution to the above system of inequalities. Von Neumann’s result in the paper was to prove that a solution exists. For this, he used fix-points techniques, and needed an extension of Brouwer’s fixpoint theorem, which he proved in the paper.

The purpose of von Neumann’s model was not to solve a concrete economic problem, but to develop economic theory. His approach was to set up an abstract structure for a general economy. His model was not evaluated against a reality outside of mathematics, but against internal mathematical consistency, namely, the existence of a solution to the system of inequalities. The (lacking) relationship between reality and the model was criticized by the economist Champernowne (1945, pp. 10-12):

Approaching these questions as a mathematician, Dr. Neumann places emphasis on rather different aspects of the problem than would an economist . . . The paper is logically complete . . . But at the same time this process of abstraction inevitably made many of his conclusions inapplicable to the real world . . .

Champernowne ends with the warning that “utmost caution is needed in drawing from them [von Neumann’s results] any conclusions about the determination of prices, production or the rate of interest in the real world” (1945, p. 15). This warning illustrates that there is not necessarily an agreement about what counts as a “solution”, when mathematics is used in other fields of inquiry – it depends on the context. It relies on the disciplinary lens used, especially when new modes of inquiry are under development. It shows how scientists from different areas might disagree about the epistemic value of model results. Despite Champernowne’s critic, von Neumann’s model has played a significant role in the development of theoretical economics (Dore et al., 1989).

Biology and dynamics from physics

Our second example is Volterra’s initial work on the Lotka-Volterra equations of predator-prey systems during the mid 1920’s. The biologist D’Ancona had compared fishery statistics from the Upper Adriatic before, during, and after the Second World War. He observed that the reduced fishing seemed to be more favourable for the predator fish than for the prey, and asked Volterra if he could explain this phenomenon (Volterra, 1926).

Volterra approached the problem in accordance with methods of classical mechanics focusing on internal causes and disregarding external ones (friction from the environment). It is not obvious how methods of mechanics can be transferred to the predator-prey system and used to grasp the underlying mechanism in the biological system. Volterra commented this by saying:

on account of its extreme complexity the question might not lend itself to a mathematical treatment, and that on the contrary mathematical methods, being too delicate, might emphasize some peculiarities and obscure some essentials of the question. To guard against this danger we must start from the hypotheses, even though they be rough and simple, and give some scheme for the phenomenon. (Volterra, 1928, p. 5)

He constructed a hypothetical system that only took the predatory and fertility of the co-existing species into account. He assumed that prey and predator increase and decrease continuously in order to be able to apply the mathematics of differential calculus, that the system is homogenous, that the birth rate of the prey is constant, and that the number of predators decreases exponentially if there is no prey it can feed on. To describe the mechanism of predation, he followed a mechanical analogy. He assumed that the number of encounters between prey and predator is proportional to the product of the numbers of the two species, as the number of collisions between particles of two gasses is proportional to the product of their densities. He assumed that the interaction between two competing species depends on the number of collisions. He called it “the method of encounters”. He let N_1 denote the number of prey, N_2 the number of predators and t the time. The above assumptions then led Volterra to the equations now known as the Lotka-Volterra equations (Volterra, 1927/1978, p. 80):

$$\frac{dN_1}{dt} = (\varepsilon_1 - \gamma_1 N_2)N_1$$

$$\frac{dN_2}{dt} = (-\varepsilon_2 - \gamma_2 N_1)N_2$$

Volterra’s mathematical analyses of the system showed its now well-known periodic cyclic behaviour of the species and confirmed D’Ancona’s observation. Volterra was concerned with the relation between the empirical data and the mathematical system. D’Ancona was not convinced that Volterra’s theory could be validated by the empirical data. This, however, did not make D’Ancona reject Volterra’s biomathematics. Israel (1993) has interpreted this as indicating a shift towards a more modern abstract modelling approach. In D’Ancona’s opinion, Volterra’s theory did not need to be supported by empirical data:

My observations [of the fisheries in the Upper Adriatic] could be interpreted in the sense of your theory, but this fact is not absolutely unquestionable: it is only an interpretation . . . You should not think that my intention is to undervalue the

experimental research supporting your theories, but I think that it is necessary to be very cautious in accepting as demonstrations these experimental researches. If we accept these results without caution we run the risk of seeing them disproved by facts. Your theory is completely untouched by this question. It lays on purely logical foundations and agrees with many well-known facts. Therefore it is a well-founded working hypothesis from which one could develop interesting researches and which stands up even if it is not supported by empirical proofs. (D'Ancona to Volterra 1935, quoted from Israel, 1993, p. 504)

In Volterra's approach, methods and theories from other fields (including mathematics) guided his construction of the equations. D'Ancona's letter shows that it can make sense and lead to new insights to investigate a mathematical model derived from a concrete phenomenon even if it cannot be confirmed by data. The system was criticised for not taking predator's adaption to new conditions into account (Knuuttila & Loettgers, 2017). This critique might relate to the crossing of disciplinary boundaries, discussions and disagreements about the relationship between reality and model, as we saw in the von Neumann case as well.

Wrong but rich explorations

Our last example is Rashevsky's attempt to derive a physical-mathematical explanation of cell division. He was a pioneer in mathematical biology (Abraham, 2004), and, even though his approach to explaining cell-division failed, his work shows ways of how scientists work, explore and think about the use of mathematics to gain insights into phenomena outside of mathematics. He explained his ideas to biologists at a symposium for quantitative biology in 1934 where he got into a debate with the biologists. His talk and the following discussion are published in Rashevsky (1934). In the introduction, Rashevsky carefully explained his scientific views, his methodology and his presumptions:

Unless we postulate some factors unknown to the inorganic physical world . . . it is simply a logical necessity, free of any hypothesis, that some physical force or forces must be active within the cell to produce a division of the latter into two or more smaller cells . . . If however we entertain the hope of finding a consistent explanation of biological phenomena in terms of physics and chemistry, this explanation must of necessity be of such a nature as the explanation of the various physical phenomena. It must follow logically and mathematically from a set of well defined general principles. (Rashevsky, 1934, p. 188)

Rashevsky argued that in order not to assume some independent mechanism "we must take some such general phenomenon [that occur in all cells] and investigate its mathematical consequences" (1934, p. 188). He chose metabolism and made an analogy of cell division to the physical phenomenon of droplets. He conceptualized a cell as a liquid system that takes in some substances from the surrounding medium and transforms them. He

treated it as “a phenomenon of diffusion governed for a quasi-stationary state by the equation $D\Delta^2c = q(x, y, z)$ where D denotes the coefficient of diffusion, c the concentration, and $q(x, y, z)$ the rate of consumption of the substance” (Rashevsky, 1934, p. 189).

At the level of molecules, he then derived expressions for the forces produced by a gradient of concentration: the force exerted on each element of volume of the solvent by the solute, the force acting on each volume as a result of osmotic pressure, and the force of repulsion between molecules. By adding these three forces, he found an expression for the force per unit volume produced by a gradient of concentration². He then made further idealizations to homogenous and spherical cells, which as he explained “will give us a general qualitative picture of various possible phenomena” (Rashevsky, 1934, p. 191).

These idealizations made it possible for Rashevsky to calculate the surface and volume tension of the cell, and he deduced that when the cell reaches a certain size, a division of the cell will result in a decrease of the free energy of the system. Based on the principle of free energy, he then concluded that his investigations had established “the necessary but not the sufficient conditions for spontaneous division” (Rashevsky, 1934, p. 192).

Rashevsky’s (1934) ambition was to build a mathematical biology similar to mathematical physics. His approach is theoretical and hypothetical, addressing causal relationships and analysing these. He drew analogies to droplets and physical liquid systems, and used differential equations. From physics, he further used notions of physical forces and the principle of free energy.

The biologists criticized Rashevsky’s modelling. They wanted to know what was “the nearest example in nature to this theoretical case” (Rashevsky, 1934, p. 195). They questioned Rashevsky’s explanation as a general solution, since “a spherical cell isn’t the commonest form of a cell”, as the biologist Davenport emphasized (p. 198). Rashevsky defended his simplification of the cell’s shape by emphasising that this systematization allowed him to investigate the liquid system and mechanisms potentially responsible for the division. While Rashevsky found it a promising result, the biologists found it irrelevant. They were not interested in imaginative explanations.

Comparing the three cases

If we compare the three cases with respect to *motivation*, *modelling strategy* and *discussion of epistemic value*, we see various differences among the three cases. If we analyse them with respect to “items” that went into their modelling constructions, we identify *analogies*, *mathematical concepts and theories*, *theoretical notions from other areas* and *empirical data*. We have summarized the differences and the similarities found in our analyses of the historical cases in Table 2.

Table 2. Illustration of our methodology and summary of the results of our analyses of the historical cases

		Von Neumann	Volterra	Rashevsky
Meta aspects	Motivation	Develop economic theory	Explain a concrete phenomenon due to reduced fishing	Explain cell division in terms of physics and chemistry
	Strategy	Abstract mathematical structure of a general economy	Simple hypotheses, simplifications and idealizations	Conceptualized a cell as a liquid system that transforms substances
	Discussion of epistemic value	Existence of solution/internal consistency vs, lack of reality/not useful	Verification through data vs. purely logical foundation	Possible explanation, promising vs. imaginary causes, irrelevant
Items used in modelling construction	Analogies		Collisions of molecules	Physical phenomenon of droplets
	Mathematical concepts / theories	Linear inequalities, fix-point techniques	Calculus, systems of differential equations	Differential equation (diffusion equation), integration
	Theoretical notions from other areas		Method of encounters	Physical forces, surface/volume tension, free energy, principle of free energy
	Empirical data		Served as motivation not as verification	Served as control not as verification
Model function	Explorative function 1., 2., 3.	2. Proof of principle	2. Proof of principle 3. Possible explanation	1. Starting point 3. Possible explanation

With regard to the “function” of the models, we use Gelfert’s work on exploratory modelling. He finds that “models . . . allow us to extrapolate beyond the actual, thereby allowing us to also explore possible, e.g. counterfactual, scenarios” (Gelfert, 2018, p. 8). He distinguishes between three functions of explorative models: 1) to find a starting point for future investigations, e.g. when an underlying theory is not known; 2) to provide proof of principles; and 3) to offer potential explanations. All three functions are present in our

historical cases. We find Rashevsky's modelling of cell division to be a prime example of explorative models, which functioned both with the aim to find a starting point, a promising way to advance in the search for the explanation of cell division, and it also managed to offer a possible explanation. Von Neumann's model can be analysed in terms of Gelfert's second function, as proof of principle provided by the mathematical so-called existence proof of the existence of equilibrium. Finally, Volterra's model (which is also used by Gelfert) is an explorative model that functioned both as a proof of principle that "differential equations is suitable for generating insights into the dynamics of (discrete) population" (Gelfert, 2018, p. 12), and it provided a potential explanation for the observed phenomenon. In Table 2, "model function" refers to these three explorative functions of Gerlfert.

Analysis II: knowledge to be taught

In Denmark, an experienced upper secondary mathematics teacher is hired by the Ministry of Education to support and guide the development of curriculum, exam exercises, etc. Together with representatives from the Ministry, this teacher manages the groups in writing the drafts for curriculum, the ministerial guidelines, and the written high stake exit examinations. Those groups consist of other experienced teachers and representatives from the noosphere. The high stake exit examination serves as entrance examination for higher education and the majority of the students are to be assessed in the written exam. Upper secondary is not mandatory in Denmark. It is a secondary education for youth, who wants to pursue higher education. Mathematics can be studied at A, B and C level, A being the most advanced. In this paper, we focus on level B, since the vast majority of students are completing mathematics at this level. There is no license for textbook writers in Denmark, and often textbooks are authored by the experienced teachers involved in reform efforts and forming the written exam. Textbooks are published by private publishers (see Jessen et al., 2019). The latest reform was implemented in 2017. Exams following the reform have been held every June, August and December since 2018. We have analysed all of the exam exercises for level B, identified exercises being related to modelling as those which tend to describe a real world situation, ignoring those requiring statistics. We have also analysed the three most commonly used textbooks (Carstensen et al., 2018; Clausen et al., 2018; Grøn et al., 2017) written on the basis of the curriculum and ministerial guidelines.

Ministerial guidelines

The ministerial guidelines state that mathematics covers a number of methods for modelling and that students should become able to:

apply functions to construct models describing data and knowledge from other disciplines, analyse mathematical models, create simulations, predictions and reflect upon the idealistic nature of models and their domains . . . including the

treatment of more complex problems. (Danish Ministry of Education, 2017, p. 1)

The curriculum is formulated in terms of competencies (called disciplinary goals) and content goals (Niss, 2018), where the latter covers: “principle properties for mathematical models with applications of the above mentioned [linear, polynomial, exponential and power] functions and combinations hereof” (Danish Ministry of Education, 2017, p. 2). Modelling is part of both content and competences students should acquire. The ministerial guidelines further state that “students are expected to apply functions for modelling purposes . . . and they are expected to gain knowledge about the phases of a modelling cycle” (Danish Ministry of Education, 2020, p. 14). The guidelines provide no explicit description of a cycle. It is required that teachers must “connect content area across domains . . . that students should learn to engage with new atypical modelling problems” (Danish Ministry of Education, 2020, p. 1). The phrase “new atypical modelling problems” is a broadening of the perspective on modelling compared to previous curricula.

Referring back to our Table 2 for analysis, the ministerial guidelines nurture the idea of students being able to combine different disciplines to solve more complex problems from the real world. The approach they suggest (without specifying) is for the students to gain knowledge about the phases of a modelling cycle. There is no mentioning of potential discussion of epistemic value, nor if analogies are relevant tools for modelling. The phrase “new atypical modelling problems” could encompass exploratory modelling for deriving possible explanations (see Table 3 ahead).

Exam exercises

As it has been stated several times during the last century, backwash of exams on classroom activities is dominant (Suurtamm et al., 2016), which is why we have analysed all level B exam exercises in the period 2018-2020. The exercises were categorised by what mathematics was needed to solve them (analytic geometry, statistics, combinatorics, etc.), and if they describe situations outside of mathematics. Exercises explicitly describing a real world phenomenon have been characterised as modelling exercises. Those often require students to carry out a regression or calculations using their CAS-tool.

An example of a modelling exercise is exercise 10, from August 2019. The students were given the expression:

$$E(x) = 0.0001 \cdot x^2 + \frac{2000}{x}, \quad 20 < x < 300$$

where $E(x)$ denotes the variable costs per good (in Dkk), when producing x goods. The students were asked to determine how many goods are to be produced to reach costs of 25 Dkk. Secondly, the students were asked to determine how many goods correspond to the lowest costs possible. The context of the problem plays no role at all, as the students are

given an expression and asked to solve $E(x) = 25$, to find $E'(x)$ and solve $E'(x) = 0$ with respect to x . The last part can also be done, if students graph the function using their CAS-tool and find the minimum of the function in the domain or use the *solve* command. Nowhere in the exam exercises from 2018-2020 are the students asked to reflect upon the model or to construct models themselves. This finding aligns with a study by Frejd (2013), where only limited elements of modelling processes are assessed in exam exercises.

The exam exercises serve the purpose of assessing students' ability to apply the notion of function, apply certain types of functions and, in some cases, basic calculus to a mathematized part of the real world. This covers the mathematical concepts and techniques used. In exercises asking the students for regressions, data are provided. However, most of our variables in the analysis is not present in the exam exercises, which might reflect the challenge of assessing more complex practices of modelling.

Textbooks and mathematical modelling

The first book of our analysis, Carstensen et al. (2018), is a grade 11 book in a series, where the grade 10 book introduces the notion of function, linear, exponential and power functions using examples where these functions describe real world "relations" as they call them, without using the notion of model. The grade 11 book has 10 chapters on mathematical domains and the last three chapters are addressing "Supplementary content", covering "Third degree polynomials", "Irrational numbers and pi" and "Mathematical models".

The authors introduce the chapter on models by stating:

When mathematical methods are applied on (more or less) realistic problems we call this **applied mathematics**, or that you create a **mathematical model**. . . We here present some examples of models, which illustrate the manifold of applications. (Carstensen et al., 2018, our translation, authors' bold face)

The first example concerns a football fan who wants to know the optimal seating in a stadium having the widest angle towards the goals. The modelling problem is addressed from two different theoretical standpoints: *the functional method* and *the geometric approach*. The students are invited to take part in solving the optimization problem. The models are given to the students as Figure 2 shows, by a sketch of the situation.

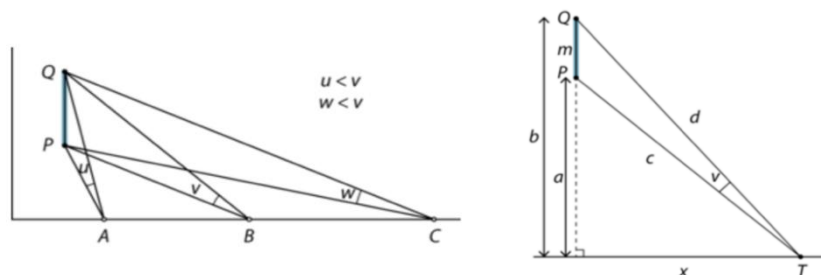


Figure 2. The sketches supporting the modelling of the optimization problem of football stadium seating in the functional approach (Carstensen et al., 2018)

The students are asked to complete six exercises, to answer the modelling problem. Here the steps of developing the model are given to them, and they are asked to check that the formulas are correct in relation to the sketches in Figure 2. One example is:

Apply the cosine relations on ΔPTQ and show, that you get:

$$\cos v = \frac{x^2 + ab}{\sqrt{(x^2 + a^2)(x^2 + b^2)}}$$

(Carstensen et al., 2018, exercise 2, chapter 13.1)

This does not give the students experience with the construction of models. The chapter further presents three other modelling problems. The first is a version of the “couch problem” (e.g., Moretti, 2002). The second problem concerns a cylinder lying down where the problem is how to measure the volume of the fluid in the cylinder only knowing the height of the fluid. The situation is depicted by a diagram, such as illustrated in Figure 2, and students are guided to an answer through seven exercises. The last problem describes “A mechanical device which can be described as a right-angled triangle” where students through nine exercises determine where to place another right-angled triangle inside of the first one fulfilling specific conditions. The real world plays no role in this problem and students get no information about the device or what it can be used for. The tasks are on pure mathematics combining geometry, algebra and differential calculus.

The book series also have a volume of exercises corresponding to the chapters of the textbooks. However the last three chapters are covered by “A mixture of exercises”. These are of the same nature as the exam exercises. In none of them are the students invited to construct models, mathematize the real world or reflect upon modelling results.

The second textbook in our analysis has a chapter on modelling (Clausen et al., 2018). The chapter begins with a narrative of applications of mathematics in other disciplines and a short presentation of an example of a modelling process. The introduction to the notion of modelling starts by: “Mankind’s desire to understand his environment serves two main purposes, firstly to explain, perhaps satisfying man’s curiosity, and then to use the knowledge to advantage” (John D. Donaldson in Clausen et al. (2018, p. 54)). A brief description of the use of models for quantitative and qualitative answers is given. A general process for modelling is outlined, as: identify patterns, choose state variables, obtain relationships connecting the variables, obtain mathematical solution and compare the solution with the physical situation (Clausen et al., 2018). It is later stated that a model can

have a limited domain but be useful for predictions. The process is depicted as shown in Figure 3.

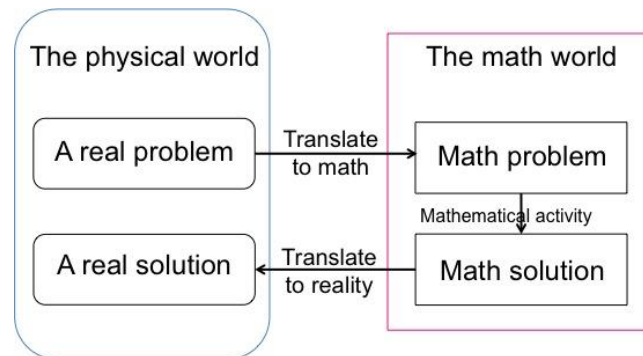


Figure 3. The modelling process as it is depicted in (Clausen et al., 2018, p. 62) as a translation from real world to mathematics and back

The examples of the book cover: folding a piece of paper, the maximum size of a barge in a canal, and how to model sea level according to tides, air pressure and height above sea level, the age of dinosaur bones and an example of assumptions made during a modelling process on optimal speed limits on main roads. The last example leads to unrealistic results. The authors seek to remedy this by asking students to redo the calculations of the model changing variables to specific numbers. None of the examples invite students to explore the models themselves (Clausen et al., 2018). The very last section of the chapter is called “More on models” and lists what can be modelled, where models are used in financial and political decision making and discuss if we can trust models, but no concrete examples are provided. The exercises connected to this chapter are similar to the example provided above from Carstensen et al. (2018).

The first chapter in our last textbook is titled “Mathematical modelling using functions” (Grøn et al., 2017, p. 26). The chapter begins by summarising the grade 10 book’s focus on variables and states that emphasis will be on representations: formula, data sets, graphs and “explained by words”. The notion of function is introduced through a description of the historic origin of optics and colours of the rainbow, including theory from physics, leading to different mathematical models for the scattering of light. The degree of openness of the examples provided in the beginning is discussed, but the examples showing students how to answer real world problems are based on existing models and formulas. The chapter ends with a list of different examples and projects related to optics, construction of a gate and optimisation of different containers. The online materials include data for planet motions, temperature fluctuations at geographical spots, population data for arctic hare and wild cats in Canada, average income in Denmark and more. Those provide an option for students to autonomously explore data and construct models.

The book summarises mathematical modelling practice as (our translation):

1. Problem formulation (delimit problem, what do we know and what do we need?)
2. Analysis and mathematical description (e.g. identify variables, relations (if you encounter expressions with roots, choose other relations))
3. Mathematical solution to problem (apply relevant mathematical methods)
4. Interpret the results (translate the mathematical result into natural language and relate to the real-world problem. (Grøn et al., 2017, p. 48)

This is similar to Figure 3 and both textbooks seem to draw on models for modelling from educational research as those discussed by Niss and Blum (2020). In general, the textbooks offer students meta knowledge regarding the importance of modelling in society and in natural sciences, and that *others* are able to go across disciplinary boundaries, when modelling real world situations. The practice students are invited to take part in is delimited to certain steps in the mathematical domain formulated as smaller tasks. The two last books provide knowledge about what has motivated certain models, where mathematics is applied to natural and social sciences. Approaches and discussion of epistemic value, nor choices of *items* in model construction are addressed in the examples. Similarly in the exercises, students are not invited to discuss epistemic value, modelling strategy, nor items employed when building models. Potentially, this might happen in the proposed projects in the last book. We have no insights into how these projects are used by teachers. This analysis is summarised in Table 3.

Table 3. Illustration of our analysis of the cases with respect to mathematical modelling in Danish upper secondary school

		Ministerial guidelines - intentions	Exam exercises	Textbooks
Meta aspects	Motivation	The importance of mathematization of the real world into mathematics is emphasised. 'connect content area across domains . . . that students should learn to engage with new atypical modelling problems', 'treatment of more complex problems' (Danish Ministry of Education, 2020, p. 1)	To assess if students are able to apply the notion of function in relation to some real world context, having no impact on the actions students need to take. Thus arithmetic, algebra, basic calculus is assessed.	Examples are provided explaining a little about the purpose and motivation for constructing certain models and apply mathematics in natural and social science. One book omits this
	Strategy	'gain knowledge about the phases of a modelling cycle' (Danish Ministry of Education, 2020, p. 14)	None	This is not developed for the cases, but two books present versions of a modelling cycle

	Discussion of epistemic value	None	Not present in the analysed items	Not addressed
Items used in modelling construction	Analogies	None	None	None
	Mathematical concepts / theories	'principle properties for mathematical models with applications of the above mentioned [linear, polynomial, exponential and power] functions and combinations hereof' (Danish Ministry of Education, 2017, p. 2)	Calculus, notion of function, different type of functions (linear, exponential, power & polynomial)	Mainly notion of functions, different types of function, differential calculus, algebra and geometry.
	Theoretical notions from other areas	knowledge from other disciplines	None	Few elements from target disciplines are included, e.g. scattering of light from physics.
	Empirical data	Describing data and knowledge from other disciplines	Exists in exercises asking students for performing regression	Present data and measures for the situations to explain existing models
Model function	Explorative function 1., 2., 3.	3. Possible explanations (though not explicitly). Other core functions of scientific models: predictions, explanation and representation. Reflect upon the idealistic nature of models and their domains	No exploration	3. Possible explanations (though not explicitly). Other core functions of scientific models: explanation of formulas from other disciplines in order to predict and explain observations

When we argue that the model function is mainly 'possible explanation', it reflects that the models presented provide some explanation of a phenomenon, e.g. the cooling of coffee, though there is no exploration since the model is already there.

Discussion and concluding remarks

To follow up on the relation, if any, between mathematical modelling in scholarly contexts and the Danish upper secondary school context, which is highly influenced by the didactical construct of the modelling cycle, we compare Table 2 and Table 3. The comparison indicates similarities and differences between scholarly knowledge and school knowledge to be taught. The curriculum emphasises the importance of being able to work across disciplinary boundaries and to apply mathematics when addressing atypical problems. Students should

recognise the role played by modelling in various contexts. These aims could nurture activities inviting the students to develop practices similar to those presented in our historic cases such as drawing on analogies, drawing on knowledge from the target discipline, developing (learning) new mathematics, and discuss the epistemic value of the model from the different disciplinary viewpoints. Further, the curriculum proposes that the approach to modelling should provide the students with knowledge about some modelling cycle. This is explicitly facilitated in two of the textbooks presenting examples of four-phased cycles as shown in Figure 3. Such descriptions of modelling and modelling competency are related to scholarly knowledge of didactics of mathematics, as, e.g., presented by Niss and Blum (2020), where the cycles are meant as analytical models, which can be used to identify and reflect upon students' development of modelling competency, as stated by Blomhøj (2011). Thus, within the noosphere, the notion of mathematical modelling has become a mixture of reasons and practices from scientific contexts as those depicted in our historic cases and modelling cycles from didactical research, where the cycles are transposed from being reflection tools for teaching and learning to be content knowledge to be taught and learned.

In the textbook examples and exercises, emphasis is put on mathematics, because the situations already are mathematised such as the folding paper exercise, the barge moving around canals, the cylinder containing some fluid, etc. We can argue that the exercises scaffold students' engagement with the phases of the modelling cycle, leaving little potential for explorations to the students and no need for understanding the situation from other disciplines. Thus, the activities offered do not capture crucial traits of the practices found in our historic cases, but rather support students' realisation of phases in the modelling cycle leaving them with limited autonomy towards the practice, but meta knowledge regarding how to think about modelling. As argued previously, the exam exercises align with findings from Frejd (2013), where few phases of the modelling cycle are assessed. On the contrary, their main purpose seems to be assessing other pieces of content knowledge. It requires a major reform of assessment systems to change this situation, as argued by Swan and Burkhardt (2012).

Our analysis shows that the knowledge to be taught reflects to a minor degree the nature of modelling and practices we found in the historic cases, and that the external didactic transposition is affected by one didactical approach to mathematical modelling resulting in a specific picture of modelling, being slightly different from modelling in the scientific context. These findings align with those presented by Wijayanti and Bosch (2018) on proportionality, that knowledge to be taught is shaped by different elements from scholarly knowledge creating hybrids of body of knowledge, not supporting the learning of practices we strive for. In contrast to Frejd and Bergsten (2016), we might argue that the historic cases can serve as inspiration for upper secondary teaching, as ways to gain further knowledge about, e.g., the exploratory nature of modelling, the role of analogies, and the

disputes model constructions might lead to. Other approaches to design of modelling activities might nurture this as well (Barquero & Jessen, 2020). In a broader perspective, our findings reflect the clashes of disciplinary ideals and domain specific practices, as, e.g., those still to be found in recent research on systems biology. Green and Andersen (2019) demonstrate the need for and importance of preparing and teaching students interdisciplinary collaboration in this field. This could be done through modelling in context across school disciplines, not just under the constraint of the mathematics curriculum. This link to the claims formulated by Lundberg and Kilhamn (2018), that we need to explicitly address the didactic transposition of key notions of curriculum when designing curriculum and other resources, if certain traits are to be kept. This holds true for modelling as well.

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Notes

¹ The list of entries in Table 1 is not exhaustive; Boumans' list also contains e.g. policy views.

² For further analyses of Rashevsky's work and/or its use in teaching see Abraham (2014), Keller (2002), Kjeldsen (2017, 2019), Shmailov (2016).

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