Teaching for mathematical competence: the different foci of modelling competency and problem solving competency

Ensinar para a competência matemática: os diferentes focos da competência de modelação e da competência de resolução de problemas

Tomas Højgaard
Danish School of Education, Aarhus University
Denmark
tomas@edu.au.dk

Abstract. In this article, I argue that it is opportune to work with students' (mathematical) modelling competency and (mathematical) problem solving competency as two essentially different learning objectives. Such a distinction can be used to facilitate communication when establishing a general agenda in and around the classroom, and especially when developing and/or selecting suitable challenges for students. I begin by outlining what characterises the two competencies. I then highlight core differences, both at an abstract and more concrete level, by analysing how various types of exemplary student tasks can be formulated. Finally, I briefly discuss some of my own experiences when using the presented analytical approach in research and development projects and point towards possible avenues for future research, development, and debate by drawing up two hypotheses concerning which types of tasks dominate compulsory mathematics education and why.
Keywords: mathematical competencies; mathematical modelling competency; mathematical problem solving competency; mathematisation competency; task design.

Resumo. Neste artigo, defendo que é oportuno trabalhar para a competência de modelação (matemática) e para a competência de resolução de problemas (matemáticos) dos alunos como dois objetivos de aprendizagem essencialmente diferentes. Tal distinção pode ser usada para facilitar a comunicação, ao criar-se uma agenda geral relativamente ao trabalho a desenvolver na sala de aula e ao redor da sala de aula, especialmente, quanto ao desenvolvimento e/ou seleção de desafios adequados para os alunos. Começo por delinear o que caracteriza cada uma das duas competências. Em seguida, realço as suas principais diferenças, tanto a um nível mais abstrato quanto a um nível mais concreto, analisando como podem ser formulados vários exemplos de tipos de tarefas para os alunos. Finalmente, discuto brevemente algumas das minhas próprias experiências de utilização da abordagem analítica apresentada, em projetos de pesquisa e desenvolvimento, e aponto possíveis caminhos para investigações futuras, desenvolvimentos e debates, elaborando duas hipóteses sobre

Quadrante: Revista de Investigação em Educação Matemática
https://doi.org/10.48489/quadrante.23691
Received: February 2021 / Accepted: October 2021

os tipos de tarefas que dominam a educação matemática na escolaridade obrigatória e o respetivo motivo.
Palavras-chave: competências matemáticas; competência de modelação matemática; competência de resolução de problemas matemáticos; competência de matematização; desenho de tarefas.

Introduction

Consider the following tasks:

1. Which means of transport is the best?
2. How is the overall tax burden affected by the rate of income tax and the rate of VAT?
3. When kicking a football, its movement can be described in terms of the trajectory for the vector-valued function $\vec{r}$, expressed as

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} k \cdot t \\ 10t - 3t^2 \end{pmatrix}, t \geq 0,$$

   where k is a constant, $0 < k < 8$, and the coordinate functions $x(t)$ and $y(t)$ indicate, respectively, the horizontal and vertical position of the ball (measured in metres) at the point in time t (measured in seconds after the kick).

   a) Calculate the football's velocity $|\vec{r}'(t)|$ at the point in time $t = 1$, when $k = 2$.
   b) Calculate the point in time at which the football first bounces.

   In a specific example, the football bounces 26 metres from the position it was kicked.

   c) Calculate k for this kick.

In this paper (which is an updated and extended version of Højgaard (2010), with contributions from Jensen (2009)), I focus on the task design aspect of teaching mathematical modelling competency and mathematical problem solving competency, with the above tasks serving as a means for discussing the crux of the former. This approach is not based on a naive assumption that task design is the only way to foster these and other competencies as part of mathematics teaching. On the contrary, I acknowledge the importance of the entire educational environment, including, for instance, establishing a constructive didactical contract (Brousseau, 1997) and supportive sociomathematical norms (Yackel & Cobb, 1996), by analysing the use of different kinds of tasks in a communicative perspective. Hence, I will elaborate on and exemplify the following conclusions:

- The reflections among, and communication between, teachers and students regarding the crux of different mathematical competencies can be utilised to facilitate the teaching of these competencies and hence their development among the students.

- One of the benefits of identifying the different foci is that doing so can help teachers hone their approaches in supporting students' development of both competencies, such as when formulating and orchestrating written tasks.

Two approaches to discussing mathematical modelling and problem solving

The teaching and learning of mathematical problem solving and mathematical modelling have long been discussed in research on mathematics education. Mathematical problem solving was among the dominant themes in the 1970s and 1980s, and when mathematical modelling gained momentum during the 1980s, not least through the establishment of the biannual conference series ICTMA, it was often with reference and links to applied mathematical problem solving (Blum & Niss, 1991; Kaiser & Brand, 2015).

As a mathematics teacher, it is easy to envisage teaching situations that challenge students to perform mathematical modelling and problem solving simultaneously, and the research literature – e.g., section 6 in Lesh et al. (2010) – offers several perspectives on such situations. Two positions seem to dominate, distinguished by the role attributed to mathematical modelling.

The first position considers mathematical modelling *a means* – a didactical tool – to achieve other learning goals. Consequently, the conceptualisations developed, and the teaching approaches tried out are discussed and refined to make mathematical modelling as effective a tool as possible in relation to specific learning goals.

An example of such a perspective is the concept of model-eliciting activities developed by Richard Lesh and colleagues (Lesh & Doerr, 2003). In their work, the primary learning goal is conceptual understanding (Lesh & Harel, 2003), approached by means of "the notion that people learn mathematics *through* problem solving and that they learn problem solving *through* creating mathematics (i.e., mathematical models)" (Lesh & Zawojevski, 2007, p. 782, italics in original). Hence, model-eliciting activities are developed to facilitate students' engagement with applied mathematical problem solving, which is found to be a motivating and effective way to learn mathematics – that is, to develop an understanding of mathematical concepts and procedures.

The second position regards the ability to carry out mathematical modelling *a goal* in itself for mathematics education. This position is in alignment with an international trend in mathematics education of prioritising student mastery over their acquisition of subject knowledge as the primary goal in national curricula. There are a wide array of reports and books describing this ambition from countries including (cf. Niss et al., 2016) Denmark (Niss & Jensen, 2002), Germany (Blum et al. 2006; Blum et al. 2015), Portugal (Abrantes, 2001), and the USA (National Research Council, 2001; NGA Center & CCSSO, 2010).

From this position, the ability to carry out mathematical modelling is considered one of several parallel learning goals related to mathematical mastery, which always also includes a goal related to mathematical problem solving. Hence, from this position, it is an important analytical task to describe and discuss what the foci of such goals should be and where they overlap. By doing so, it is possible to strike a balance, ensuring the goals are relevant from a mastery perspective while being teachable at a classroom level. This article is my attempt to carry out such an analysis of goals related to mathematical modelling and mathematical problem solving.

A competency perspective

My work with mathematical modelling and mathematical problem solving in relation to mathematics education contains an analytical bias. This bias stems from my involvement in the development of a more general principle: The mastering of mathematics can be understood in terms of the development of a set of mathematical competencies that can and should be taken into account within mathematics education.

The scaffolding of this idea was the hub of the so-called KOM Project (KOM is an acronym for "competencies and mathematics learning" in Danish), conducted in the years 2000-2002 under the leadership of Mogens Niss from Roskilde University in Denmark, for which I (then with the name Tomas Højgaard Jensen) was the scientific secretary. The project and its findings are thoroughly reported in Niss and Jensen (2002), with Niss and Højgaard (2019) presenting an updated extract in English, while Niss and Højgaard (forthcoming) includes an English translation of much of the original report. The key steps in the project's analysis were to

- move from a general understanding of the concept *competence*, which I – not far from the conceptualisation in the KOM Project – understand as someone's insightful readiness to act in response to the challenges of a given situation (Blomhøj & Jensen, 2007),
- to a focus on *a mathematical competency*, defined as someone's insightful readiness to act in response to a certain kind of mathematical challenge of a given situation (Ibid.),
- and then identify, explicitly formulate and exemplify *a set of mathematical competencies* that can be agreed upon as independent dimensions spanning what it means to master mathematics.

The results of the KOM analysis are visualised in condensed form in Figure 1. From a curriculum perspective, such a set of mathematical competencies has the potential to replace the syllabus as the hub of development within mathematics education, offering a vocabulary for a focused discussion of the aims of mathematics education that can make us feel comfortably scaffolded for the same reasons that we are currently comfortable with the traditional specificity of a syllabus-driven curriculum (Blomhøj & Jensen, 2007; Højgaard & Sølberg, 2019).

In this article, I elaborate on a specific part of the KOM perspective: the idea that mathematical modelling competency and mathematical problem solving competency are two distinct, but overlapping, constituents of mathematical mastery. In doing so, I make use of quotes from a translation of much of the original KOM report to be published in Niss and Højgaard (forthcoming) and from the updated description of the KOM framework in Niss and Højgaard (2019).

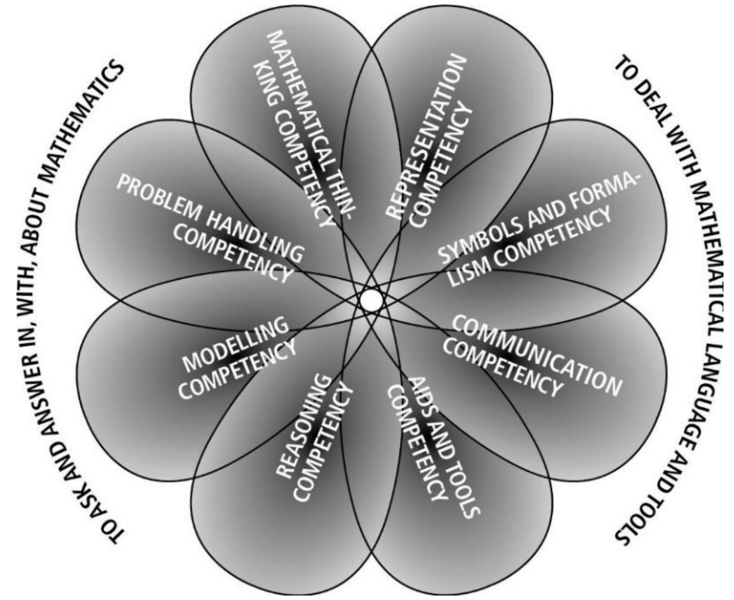

Figure 1. A visual representation – the "KOM flower" – of the eight mathematical competencies presented and exemplified in the KOM report (Niss & Jensen, 2002, p. 45; Niss & Højgaard, 2019, p. 19)

Mathematical modelling competency

In brief and simple terms, this competency concerns the ability to create and deal with mathematical descriptions of something that is not intrinsically mathematical. More precisely, I adopt a holistic conceptualisation approach (Shavelson, 2010) and use *mathematical modelling competency* to describe someone's insightful readiness to carry out all steps in a mathematical modelling process in a specific context and to critically analyse mathematical models produced by others (Blomhøj & Jensen, 2003; Jensen, 2007a).

We provided the following longer and more detailed description of this competency in the updated outline of the competencies identified in the KOM Project (Niss & Højgaard, 2019, p. 16):

> This competency focuses on mathematical models and modelling, i.e., on mathematics being put to use to deal with extra-mathematical questions, contexts and situations. Being able to construct such mathematical models, as well as to critically analyse and evaluate existing or proposed models, whilst taking purposes, data, facts, features and properties of the extra-mathematical domain being modelled into account, are the core of this competency. It involves relating to and navigating within and across the key processes of the "modelling cycle" in its various manifestations . . .

There are two aspects that it is worth highlighting in these descriptions. Firstly, the competency – like its "siblings" in the KOM report – has both a receptive and analytic facet, focusing on understanding and critically assessing existing processes (similar to what Greer and Verschaffel (2007) classify as critical modelling), and a constructive facet, focusing on the ability to conduct such processes oneself, in this case mathematical modelling (cf. Niss & Jensen, 2002; Niss & Højgaard, 2019).

Secondly, these and other descriptions of competencies do not in earnest gain value and substance until one – as in the second part of the passage quoted above – ventures to describe one's understanding of the term(s) that, in purely linguistic terms, constitutes the root in terms of the naming and characterisation of the competency. If all communication regarding the content of mathematical competencies is duty bound to provide such a conceptual clarification, the competency approach maintains its value as a source of constructive debate. Analyses such as those found in the KOM report are to be read as an invitation to such debate, not as a simple incantation of a set of canonical concepts that have awaited definitive clarification and subsequent memorisation.

The mathematical modelling process

In light of the above, I "owe" readers to describe what I mean in the above description of the mathematical modelling competency when I refer to "a mathematical modelling process". For me, in keeping with the quoted description from the KOM report, this denotes a complex and often not especially streamlined process involving a wide range of ways of thinking and acting. I have often benefitted from working with a model that describes this process with the help of the following six phases, which, if need be, are repeated (several times around the cycle), cf. the visualisation in Figure 2 (Blomhøj & Jensen, 2003, 2007):

a) Clarification of your motivation guiding the identification of the characteristics of the perceived reality you want to model.

b) Selection of relevant objects, relations, etc. within the resulting domain of inquiry, and idealisation of these in order to allow a mathematical representation.

c) Translation of these objects and relations to mathematics.

d) Use of mathematical methods to produce mathematical results and conclusions.

e) Interpretation of these results and conclusions in relation to the initial domain of inquiry.

f) Evaluation of the validity of the model through comparison with observed or predicted data or with theory-based knowledge.

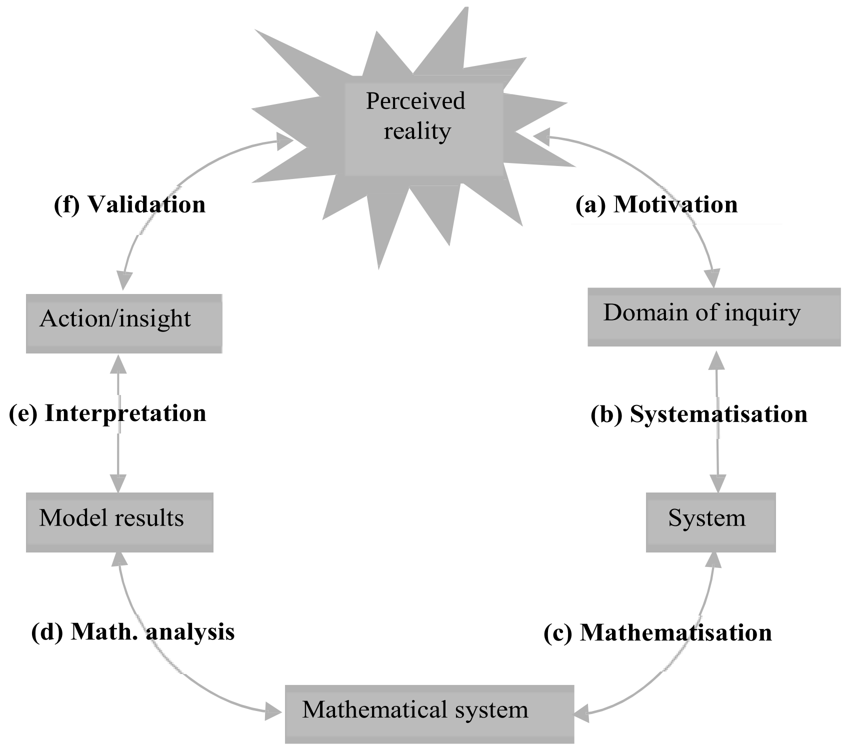

Figure 2. A visual model of the mathematical modelling process (adapted from Blomhøj and Jensen, 2007)

As well as presenting the various phases in a way that hopefully provides a useful overview of the overall process, I have also taken pause after each activity and tried to assess at what level of the modelling process it takes place.

This six-phase description is similar to, and is indeed inspired by, many of the other models of the mathematical modelling process found in the research literature on mathematics education (e.g., Blum & Leiß, 2007; Niss, 2010; Niss & Blum, 2020). The various models represent different conceptualisations of the terms model and modelling, but they all share the idea that mathematical modelling is a cyclical and iterative process of constructing and adjusting mathematical models of extra-mathematical situations to understand and explain selected phenomena (Stillman et al., 2015). Moreover, their very nature as models of the complex process of mathematical modelling means that they all intrinsically favour descriptions of sub-abilities of mathematical modelling competency (Kaiser & Brand, 2015).

The model in Figure 2, and all the concepts it contains, have been exemplified at the level of higher education in Blomhøj and Jensen (2003). In this article, I complement this with an example at the lower secondary level. However, I will first introduce the concept of mathematical problem solving competency so that this example can be used to discuss the differences at the core of the two competencies.

Mathematical problem solving competency

In brief and simple terms, this competency concerns the ability to deal with situations requiring the identification of approaches that are not immediately apparent.

More precisely, I use *mathematical problem solving competency* to describe someone's insightful readiness to solve different kinds of problems with a mathematical content (which might not be visible at first sight) that have already been formulated and to critically analyse approaches put forward by others (Jensen, 2007a).

In the KOM framework, this competency is combined with the ability to *pose* mathematical problems. As shown in Figure 1, it is termed problem handling competency, which is described as follows:

> This competency involves being able to pose (i.e., identify, delineate, specify and formulate) and to solve different kinds of mathematical problems within and across a variety of mathematical domains, as well as being able to critically analyse and evaluate one's own and others' attempted problem solutions. A key aspect of this competency is the ability to devise and implement strategies to solve mathematical problems.
>
> *Remarks*. It is inherent in the notion of "problem" that it requires more than the immediate employment of approaches, methods and procedures that are routine to the problem solver. Since what to one person is a problem may sometimes be a standard task to someone else, the notion of problem is relative to the person attempting to solve it … (Niss & Højgaard, 2019, p. 15)

The key concept that must be grasped in relation to this competency is the notion of "problem". I have found that this is among the concepts generating the greatest uncertainty when organising courses and workshops centred on descriptions of mathematical competencies for various groups of mathematics teachers. Therefore, I will now outline my understanding of the notion of "problem" in greater detail (cf. Jensen, 2007a).

Task, exercise, and problem

I use *task* as a general term for something explicitly formulated as a challenge, as opposed to a challenge that is not explicitly formulated as such and is therefore only a challenge due to someone's interpretation of the situation. A task is thus of an objective nature, in the sense that whether or not something is a task is not dependent on the person giving or receiving it.

I use *problem* to denote a situation involving a number of methodologically open questions that pose an intellectual challenge for someone who does not have direct access to methods/procedures/algorithms that would enable them to answer these questions (cf. Blum & Niss, 1991). A problem is thus of a subjective nature in the sense that, as stated above, it concerns something that "requires more than the immediate employment of approaches, methods and procedures that are routine to the problem solver" (p. 37). The existence of a problem thereby also implies the existence of one or more people for whom it is a problem.

An example of a task might be "mow the lawn" or "find the roots in the quadratic equation $2x^2 + 2x - 4 = 0$". Of course, when completing such tasks, various problems can arise. For instance, the lawnmower might be broken, one might not be able to find the correct formula or one might not have the necessary level of knowledge or experience (for an 8-year-old child, both tasks would presumably be problematic). As such, tasks can be given to anyone, but it is not always possible to know for certain for whom it will be a problem.

In order to be able to clearly distinguish between the terms, I refer to *an exercise* when it is reasonable to expect that a task is not and will not cause a problem for the recipient. In the cases where solving a task causes a problem for the recipient, I use the term "problem" instead of task. The term "task" therefore constitutes the union of the terms exercise and problem and is used when it is not possible to determine the recipient's capability or when the distinction between exercise and problem is not important.

Mathematical problem solving

Problem solving is used, plain and simple, to denote the process of trying to solve a problem. Crucial to this process is that – as the "complement" to working with exercises – it is characterised by the necessity of conscious or unconscious reflection regarding method. The mathematics education literature on problem solving – e.g., Schoenfeld (1992), which is a classic in this field – contains many good suggestions regarding what, more precisely, such reflection might involve and what other challenges might arise during problem solving processes. Here, I merely highlight reflections on method to support a clarification of the concept.

I only consider it meaningful to talk about *mathematical problem solving* if that which defines the process – the reflection regarding method – involves mathematical terms, methods, and results. In other words, it is not an example of *mathematical* problem solving if mathematics is not part of the process until the point at which the problem has been successfully translated into a routine exercise.

An important consequence for teachers in this regard is that it is not sufficient to simply observe whether or not students have used mathematics in their response to a task when determining if it is an example of mathematical problem solving. It is necessary to more closely examine the response and the underlying work process to determine the way the student has used mathematics.

Contrasting the foci of the competencies

From the conceptual point of view described above, conducting a mathematical modelling process will often entail solving one or more mathematical problems, not least during the mathematisation and mathematical analysis phases (phases c and d in Figure 2). Furthermore, all applied mathematical problem solving is part of a mathematical modelling process that the recipient of the problem might only be introduced to at a stage where parts of the process have already been completed (Niss & Højgaard, 2019).

Meanwhile, there are key differences at the core of the two competencies – differences I will now further explore by discussing the nature of different types of challenge and the relevance of different types of tasks in setting the scene for these challenges.

Problem solving and a sense of methodologically being lost

Mathematical problem solving competency concerns the ability to cope with what can be characterised as a feeling of "knowing what the goal is without knowing how to achieve it" (Blomhøj & Jensen, 2003, p. 127). This ability to cope with what can be a quite frustrating sense of being cognitively stuck is, in my view, the crux of mathematical problem solving competency (Jensen, 2007a).

As an example, I pose the following task, borrowed from Schoenfeld (1985), which is a pioneering classic in instructional approaches to mathematical problem solving:

> Any triangle can be divided in two equal areas by constructing a line parallel to one side of the triangle. Specify such a construction.

As you are reading this article, you most likely have a professional interest in mathematics education. As such, I expect you see this task as a mathematical problem, inviting mathematical problem solving, because:

- In the concrete situation, you understand the task and have a clear idea what the challenge is,
- realise that there must be a reasonably clear and identifiable solution, but
- do not immediately know how to arrive at this solution.

As part of the research and development project *KOMPIS* (Højgaard & Sølberg, 2019), I developed the visual model depicted in figure 3 of these three characteristics of a situation that is well suited to developing someone's problem solving competency. The arrows represent different methods – approaches – for finding a solution to a given task. As opposed to a problem, an exercise is in this model represented by just one arrow – the method to be practised – leading straight from a task to its solution, cf. the previous distinction between task, exercise, and problem.

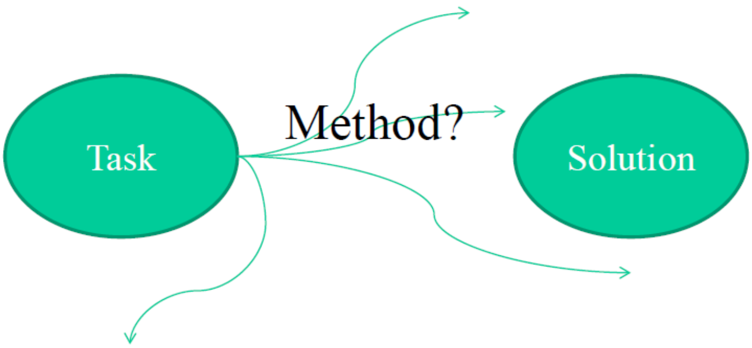

Figure 3. A visual model of a situation well suited to developing someone's problem solving competency

Modelling and dealing with openness

Mathematical modelling competency entails a working process characterised by the need for various forms of demarcation and clarification in order to meet the definition: enabling mathematical representation and analysis of an extra-mathematical challenge. The need for demarcation arises, not least, during the introductory "outer" stages of the modelling process, which involve many more concerns than those of a strictly mathematical nature. These stages correspond with the phases labelled *motivation* and *systematisation* in Figure 2.

By virtue of the "underdetermined" nature of these initial stages of the mathematical modelling process, the crux of this challenge is to learn how to deal with the many choices that need to be made (where there is often more than one sensible option) before mathematical concepts and techniques can be applied, as well as the lack of a clearly defined strategy that can be used when making these choices. Or – to put it a different way – to cope with a feeling of "perplexity due to too many roads to take and no compass given" (Blomhøj & Jensen, 2003, p. 127). Seen through a didactical "competency lens", mathematical modelling is mainly of interest if it provides a way of coping with such "openness":

> Even though in principle we are concerned with mathematical modelling every time mathematics is applied outside its own domain, here we use the terms model and modelling in those situations where there is a non-evident cutting out of the modelled situation that implies decisions, assumptions, and the collection of information and data, etc.
>
> Dealing with mathematics-laden problems which do not seriously require working with elements from reality belongs to the above-mentioned problem handling competency. Also those aspects of the modelling process that concentrate on working within the models are closely linked to the abovementioned problem handling competency. However, the modelling competency also consists of other elements which are not primarily of a purely mathematical nature, for example

> knowledge of non-mathematical facts and considerations as well as decisions regarding the model's purpose, suitability, relevance to the initiating questions, etc. (Niss & Jensen, 2002, pp. 52-53; translation from Niss & Højgaard, forthcoming)

An illustrative example

I will illustrate the didactic potential of working with the mathematical modelling competency by, as announced, presenting an example. This example is in the form of a constructed episode (Blomhøj, 2006) where a number of students, demonstratively and explicitly, are working with the motivation and systematisation phases of a mathematical modelling process, as described in Figure 2. Their work is based on the first task mentioned in the introduction to this article:

> Which means of transport is the best?

If we let the two Year 9 students, Adam (A) and Beth (B), exercise their well-developed mathematical modelling competency to solve this task, the following is an example of a possible process:

A: Out of the various options the teacher has given for our modelling project, I think we should work on the one concerning means of transport. I myself have sometimes wondered which means of transport is best. Let's explore this question with the aid of mathematical modelling.

B: That is not a straightforward question to answer – it depends what you mean by "means of transport" and "best".

A: I'm thinking of when I need to get to school. I can choose to walk, bike, take the bus or get my mum to give me a lift. Couldn't we compare these four means of transport?

B: Well, I guess, but it's still too vague to ask what's "best": The car is probably quickest and most comfortable, but you can meet more new, interesting people in the bus. Meanwhile, cycling gives the most exercise and fresh air, while walking is probably the most environmentally friendly choice.

A: The whole environmental thing is something I think about when considering which means of transport I should choose. I usually take the bus – why do you think it's most environmentally friendly to walk?

B: I'm not sure that I do think that – it was just an example! But we can decide to explore that: "What is the most environmentally friendly means of transport?"

A: No, that won't work – "environmentally friendly" is just as vague and individual a concept as "best". But we could look at the energy consumption…

Adam and Beth decide to analyse energy consumption in relation to various constructed but, for them, realistic scenarios each focused on a different means of transport. The energy consumption related to the car option causes some difficulty as it depends on whether the car would make the same journey anyway. If so, it would seem evident that there is no additional energy consumption. If not, the system to be mathematised (cf. Figure 2) is limited to a simple matter of fuel consumption.

They also have difficulties mathematising the bus scenario due to issues related to getting on and off and initially decide not to include this option in their model. This is followed by a healthy portion of mathematical analysis of the data they have produced and the emergence of a number of model findings that they now need to interpret as part of a broader evaluation of the modelling process.

B: Even though our model calculations focused on energy consumption show that cycling is the best means of transport, I would still usually choose to take the bus to school. I need to travel at least 10 kilometres each way, so I get tired just thinking about cycling, while the bus is somewhere I can sit and unwind.

A: I don't blame you. I only have about a kilometre to school, so for me, just the wait at the bus stop means it's quicker for me to cycle. I wouldn't even take the bus if it did not consume any energy at all, so our modelling is not sufficient grounds for choosing a particular means of transport.

B: As we are both so concerned with how long the journey takes, maybe we should try to draw up a model where we limit ourselves to travel time as a criterion for determining the best means of transport. In our cases, I expect the answer will be something involving a function of how far you need to travel.

A: Is it not inevitable that such a model would end up having the same problem as before? Travel time alone is an unrealistically simple basis for choosing a particular means of transport.

B: Yes, of course, but before we managed to make a good comparison by assuming that energy consumption is the only important factor, even though that is obviously not realistic. Surely, we can do that again and then discuss afterwards how the two models can be combined.

A: I think we would be better off trying to model a situation that includes both energy consumption and travel time in the system we decide to focus on from the get-go. That seems more realistic.

B: Yes, I think so too, but I also think it will be more confusing and therefore less helpful in gaining an overview of the issue, and I'm also not sure whether we will be able to juggle the mathematics that would be needed to mathematise that system you talk about. But we could give both approaches a shot…

This constructed episode allows me to highlight three points based on the work process of Adam and Beth. *The first point* concerns their dialogue itself: working with such episodes, in general, and grappling with their construction, in particular, has great potential when using descriptions of mathematical competencies as tools for didactic communication. Colleagues and I have used this approach in various contexts, with our experiences indicating two potentially beneficial factors in particular. One is that the episodic approach leads to a focus on what competency descriptions, by their very nature, are all about: an individual or individuals who act – discuss and/or exhibit activity in some other way – in response to certain challenges in a concrete situation. Another beneficial factor is that a concrete link is established between a number of learning objectives that are not always easily accessible and real-world classroom practice – a link that is always binding and therefore develops the understanding of both objectives and practice.

The second point is that the task with different means of transport, for me as a competency-oriented mathematics teacher, has the potential to invite students to work with all phases of a mathematical modelling process and thereby the potential to develop the mathematical modelling competency of all involved with a full *degree of coverage*, meaning the "*aspects* of the competency someone can activate and the degree of *autonomy* with which this activation takes place" (Jensen, 2007b, p. 143 – also see Niss & Jensen, 2002; Niss & Højgaard, 2019). This is because it seems to me to be natural to use this task as a jumping-off point for a process like the one I constructed above.

In *Appendix A,* I have provided other examples of tasks that I have found have this potential. Anyone is welcome to try using these examples themselves based on the construction of new episodes from the classroom. The examples are categorised with lesson planning in mind (cf. Jensen, 2007a): the examples in the first category are developed and tested as suggestions that can be used to inspire modelling-oriented project work where students work with the same topic for a prolonged period using inquiry-based approaches (Artigue & Blomhøj, 2013). In my experience, for the youngest students 2-4 lessons is as much as they can handle, while secondary students can benefit from more prolonged periods of project work of up to 2-4 weeks. The examples in the second category are developed and tested as suggestions for short-duration "desk-based" tasks that can function within the framework of a single lesson.

The third point is that neither task 2 nor task 3 in the introduction to this article have the same potential to develop the recipient's mathematical modelling competency with a full degree of coverage. This is partly because it requires a particularly vivid imagination to envisage a modelling process such as the one described above taking place based on these tasks. This would require that the tasks be rephrased to such an extent that they would in fact constitute new tasks, with a different focus and different learning objectives.

Mathematisation competency

The second task from the introduction – How is the overall tax burden affected by the rate of income tax and the rate of VAT? – is an example of a task that "concentrates on working within the [mathematical] models" (cf. the previously quoted passage from the KOM report about the difference between the modelling and problem solving competencies). It represents a kind of task that only challenges the student to work with phases (c), (d) and (e) of the mathematical modelling process as it is characterised here. The delimitation of the context and task in phases (a) and (b) is already dealt with in the formulation of the task, and the inclination to work with phase (f) often comes from having worked with phases (a) and (b) (Christiansen, 2001).

Tasks like this, which I consider to be similar to the model-eliciting activities mentioned above (Lesh & Zawojewski, 2007), challenge the student's competency to mathematise a more or less well-defined problem of a non-mathematical character. Hence, following the conceptualisation in this article, such tasks are mainly to be seen as an invitation to develop problem solving competency within the domain of applied mathematics. They do not challenge – and therefore cannot be used to develop – mathematical modelling competency with full degree of coverage, but properly formulated and orchestrated, such tasks can challenge what is often a vital part of developing this competency: applied mathematical problem solving focused on mathematisation.

I will label the objective of such challenges *mathematisation competency* to describe someone's insightful readiness to solve problems defined as such by a challenge to mathematise. More loosely speaking, mathematisation competency is the combination of mathematical problem solving competency and mathematisation. Appendix A presents further examples of tasks that I have found can be used to support the development of this competency.

This conceptualisation is, not least, in line with a German approach that characterises mathematical modelling competency using detailed descriptions of sub-competencies developed during the various phases of the modelling cycle (Kaiser & Brand, 2015). It deviates from the conceptualisation used in more recent developments of the PISA framework, where modelling and mathematisation are used synonymously (Turner et al., 2015).

Spanning the competencies

Table 1 is an attempt at a comparative didactic spanning of mathematical modelling competency and mathematical problem solving competency based on the conceptualisations and understandings presented in this article. Each of the tasks in the table has been chosen as exemplary for the category it represents, similar to the use of the competency framework for task design in PISA (OECD, 1999; Turner et al., 2015).

Table 1. Examples of tasks spanning mathematical modelling competency and mathematical problem solving competency

Invitations to…	***Mathematical problem solving***	***Mathematical exercises***	***Neither problem solving nor exercises***
Mathematical modelling	1. Which means of transport is the best?	2. How much fabric does one need to make a tablecloth for the dining table?	Irrelevant category
Authentic mathema-tisation	3. How is the overall tax burden affected by the rate of income tax and the rate of VAT?	4. Draw a floor plan of a 135 m^2 house.	Irrelevant category
Pseudo extra-mathematical orientation	5. The total length of the Loch Ness monster is 40 metres plus half its own length. How long is the monster?	6. Anna and Bob earn 20% from the sale of confectionery. How much do they earn if they sell for a) DKK 100? b) DKK 500?	9. When kicking a football, its movement can be described in terms of the trajectory for the vector-valued function $\vec{r}$, expressed as $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} k \cdot t \\ 10t - 3t^2 \end{pmatrix}, t \geq 0,$ [… see the introduction of the article]
No extra-mathematical orientation	7. A cube's volume is *k* times as big as the volume of another cube. What is the relation between the surface areas of the two cubes?	8. Solve the equations: a) $7 - x/3 = x + 1$ b) $x - 2 = \frac{x+2}{2}$ c) …	10. In a system of coordinates, a parabola *P* and a line *l* are determined by $P\colon\ y = x^2 - 4x + 3$ $l\colon\ y = -x + b$ where *b* is a number. Determine the coordinates of the vertex *T* of the parabola *P*. Calculate the distance from *T* to *l* for $b = -2$.

Task 1-8 of the table were developed and used in research and development projects focusing on mathematical classroom practices, not least the previously mentioned KOMPIS Project (Højgaard & Sølberg, 2019) and the so-called Allerød Project (Jensen, 2007a) - hence the overlap with the examples in Appendix A of tasks that I have actually used. Since these classroom experiments took place in grades 8-11, my examples of problems versus exercises refer to reactions from students from these grade levels: Task 1, 3, 5 and 7 are classified as invitations to mathematical problem solving because the students in the classrooms I visited generally considered them as problems in the sense described in this article, and similarly regarding task 2, 4, 6 and 8 considered as exercises. Students from lover grades might experience all of them as problems, and to tertiary mathematics students they would probably all be considered as exercises.

It would be useful to develop a comparison such as in Table 1 based on a range of constructed episodes. My reasons for not doing so are partly a lack of space and partly that I have found there are benefits to one-sided presentations that can function as an implicit call for further collective exploration in educational contexts.

Some experiences of putting the competency descriptions to work

In 2009, the set of mathematical competencies from the KOM report (Niss & Jensen, 2002) became a fundamental part of the curriculum for compulsory mathematics education in Denmark (Undervisningsministeriet, 2009) – and this remains the case in the current curriculum (Børne- og Undervisningsministeriet, 2019). This process of curricular development is thoroughly described in Niss and Højgaard (forthcoming).

During the same period of almost 20 years since the report's publication, colleagues and I have developed the conceptualisations and approaches laid out in this article and attempted to put them to work in various settings, including the structuring and writing of a series of mathematics textbooks for primary and lower secondary education (cf. Højgaard, 2019 – Gregersen et al. (2016) is a concrete example), development-oriented research projects (Jensen, 2007a; Højgaard & Sølberg, 2019) and in-service teacher education (Højgaard & Winther, 2021).

To cut a long story short, I have two main experiences from initiating and being heavily involved in these projects. Firstly, one of the main advantages of using a competency perspective in relation to mathematics education is that reflections concerning the foci of different mathematical competencies can be used to engender the kinds of work processes mathematics education seeks to develop. More specifically, it became an important and shared part of the thoroughly developed classroom culture in the development-oriented research projects to distinguish between and work with the different kinds of tasks exemplified in Appendix A, and to value their different contributions to the development of students' mathematical modelling competency. In one important example related to the study described by Kaiser and Brand (2015), the conceptualisation of mathematisation competency as being different from but explicitly related to mathematical modelling competency has been used to study combinations of holistic and atomistic approaches to the teaching of mathematical modelling (Blomhøj & Jensen, 2003).

Secondly, a wholehearted attempt to support students' development of mathematical competencies in general, and mathematical modelling competency in particular, is a both extremely meaningful and highly complex and demanding challenge for all parties. Every attempt to tackle such a challenge will therefore entail a balance between enthusiasm, driven by a sense of meaningfulness, and apathy, driven by a sense of a lack of both time and mental resources. It is both to be expected and in keeping with my experiences that various forms of support and encouragement for teaching that focuses on students' mathematical competencies tip this balance towards enthusiasm and away from apathy. Meanwhile, the opposite is to be expected if teachers do not feel they receive such support and encouragement.

Hypotheses to promote further debate and investigations

I have not tried to take a systematic approach in assessing the "health" of mathematics education regarding this balance between enthusiasm and apathy. Nevertheless, I have two experience-based – and unfortunately not especially optimistic – hypotheses regarding the use of the various types of tasks outlined in Table 1 that I hope can promote further debate and development.

The first hypothesis is that invitations to mathematical modelling are all too often replaced with invitations to mathematisation – because it is easier to formulate and orchestrate mathematisation tasks as a teacher, easier to understand such tasks as a student and easier for teachers and the educational system to incorporate such tasks in examinations due to their less open-ended nature.

The second hypothesis is that invitations to mathematisation are all too often replaced with pseudo extra-mathematically oriented tasks that are not focused on either problem solving or exercises (cf. Table 1) – because it is easier to formulate, orchestrate, work with, and assess the latter type of task. The third task from the introduction is an example of such a task (cf. Table 1). It also serves as a good illustration of the danger I hypothesise about here, since it is an authentic example (from the national exam in May 2020) of the only kind of application-oriented tasks found in the written mathematics exam for students at the highest level of upper secondary school (*gymnasium*) in Denmark.

What alternatives might there be to this type of exam question? This is an extremely relevant and challenging issue which deserves separate analysis and discussion. For now, I will merely offer one concrete suggestion from the Allerød Project: In Appendix A, all the examples of short-duration tasks focusing on mathematical modelling and mathematisation are taken from the written end-of-term examination, midterm examinations and final written examination at upper secondary level (cf. Jensen, 2007a, appendix E), which were developed and used as part of the experimental setup. There are many challenges in finding ways of assessing students' work in solving such tasks (ibid), but imagine how much more powerful a signal it would send in terms of taking mathematical modelling competency seriously as an agenda-setting fulcrum of mathematics education if such tasks set the tone for centrally developed written examinations in mathematics!

References

Abrantes, P. (2001). Mathematical competencies for all: Options, implications and obstacles. *Educational Studies in Mathematics*, *47*(2), 125-143. https://doi.org/10.1023/A:1014589220323

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM*, 45(6), 797-810. https://doi.org/10.1007/s11858-013-0506-6

Blomhøj, M. (2006). Konstruktion af episoder som forskningsmetode – læringsmuligheder i IT-støttet matematikundervisning [Construction of episodes as a research method – opportunities to learning in an IT-facilitated mathematics teaching]. In O. Skovsmose, & M. Blomhøj (Eds.), *Kunne det tænkes? – om matematiklæring* [Is it possible? – on learning mathematics] (pp. 228-254. Malling Beck.

Blomhøj, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: Conceptual clarification and educational planning. *Teaching Mathematics and its Applications*, *22*, 123-139. https://doi.org/10.1093/teamat/22.3.123

Blomhøj, M., & Jensen, T. H. (2007). What's all the fuss about competencies? Experiences with using a competence perspective on mathematics education to develop the teaching of mathematical modelling. In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Applications and modelling in mathematics education: The 14th IMCI Study* (pp. 45-56). Springer.

Blum, W., Drüke-Noe, C., Hartung, R., & Köller, O. (Eds.). (2006). *Bildungsstandards Mathematik: konkret*. Cornelsen-Scriptor.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. L. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling (ICTMA 12): Education, engineering and economics* (pp. 222–231). Horwood.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – State, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, 22, 37-68. https://doi.org/10.1007/BF00302716

Blum, W., Vogel, S., Drüke-Noe, C., & Roppelt, A. (Eds.). (2015). *Bildungsstandards aktuell: Mathematik in der sekundarstufe II*. Schrödel.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Kluwer.

Børne- og Undervisningsministeriet (2019). *Matematik – Fælles Mål – 2019* [Mathematics – Common Objectives – 2019]. Copenhagen, Denmark: Ministry of Education. www.emu.dk

Christiansen, I. M. (2001). Critical evaluation of models in relation to the modelling process. In J. F. Matos, W. Blum, K. Houston, & S. P. Carreira (Eds.), *Modelling and mathematics education: ICTMA 9: Applications in science and technology* (pp. 391-400). Horwood.

Greer, B., & Verschaffel, L. (2007). Modelling competencies – Overview. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI Study* (pp. 219-224). Springer.

Gregersen, P., Højgaard, T., Larsen, L. H., Pedersen, B. B., & Thorbjørnsen, H. (2016). *Matematrix 9*. 2. Ed. Alinea. Mathematics textbook for grade 9.

Højgaard, T. (2010). Communication: The essential difference between mathematical modeling and problem solving. In R. Lesh, P. Galbraith, C. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 255-264). Springer. https://doi.org/10.1007/978-1-4419-0561-1_22

Højgaard, T. (2019). Competencies and textbook development: A three-dimensional content model enacted in the Danish textbook series *Matematrix* for Grades k-9. In S. Rezat, L. Fan, M. Hattermann, J. Schumacher, H. Wuschke (Eds.), *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 197-202). Universitätsbibliothek Paderborn.

Højgaard, T., & Sølberg, J. (2019). Competencies and curricula: Two case stories of two-dimensional curriculum development. *European Journal of Science and Mathematics Education*, *7*(1), 50-60. https://doi.org/10.30935/scimath/9533

Højgaard, T., & Winther, N. (2021). Facilitering af kompetencemålstyret matematikundervisning: Erfaringer med kommunalt forankret, skolebaseret udvikling af lærerkompetencer [Facilitating mathematics education guided by competency objectives: Experiences with municipality anchored, school-based development of teacher competencies]. *MONA*, *1*, 50-68.

Jensen, T. H. (2007a). *Udvikling af matematisk modelleringskompetence som matematikundervisningens omdrejningspunkt – hvorfor ikke?* [Developing mathematical modelling competency as the hub of mathematics education – why not?] (Ph.D. Dissertation). Roskilde University, Denmark. https://rucforsk.ruc.dk/ws/portalfiles/portal/2051535/IMFUFA_458.pdf

Jensen, T. H. (2007b). Assessing mathematical modelling competency. In C. Haines, P. L. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling (ICTMA 12): Education, engineering and economics* (pp. 141-148). Horwood.

Jensen, T. H. (2009). *Modellering versus problemløsning – om kompetencebeskrivelser som kommunikationsværktøj* [*Modelling versus problem solving – on description of competencies as a communicative tool*]. *MONA*, 2, 37-54.

Kaiser, G., & Brand, S. (2015). Modelling competencies: Past development and further perspectives. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 129-149). Springer. https://doi.org/10.1007/978-3-319-18272-8_10

Lesh, R., & Doerr, H. M. (Eds.) (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Lawrence Erlbaum.

Lesh, R., Galbraith, P., Haines, C., & Hurford, A. (Eds.) (2010). *Modeling students' mathematical modeling competencies*. Springer.

Lesh, R., & Harel, G. (2003). Problem solving, modeling, and local conceptual development. *Mathematical Thinking and Learning*, *5*(2-3), 157-189. https://doi.org/10.1080/10986065.2003.9679998

Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester, Jr. (Ed.) (2007), *Second handbook of research on mathematics teaching and learning* (pp. 763-804). Information Age.

National Research Council (2001). Adding it up: Helping children learn mathematics. National Academy Press. https://doi.org/10.17226/9822

NGA Center & CCSSO (2010). *Common core state standards for mathematics*. NGA Center & CCSSO (National Governors Association Center for Best Practices & Council of Chief State School Officers). www.corestandards.org

Niss, M. (2010). Modelling a crucial aspect of students' mathematical modeling. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 43–59). Springer. https://doi.org/10.1007/978-1-4419-0561-1_4

Niss, M., & Blum, W. (2020). *The learning and teaching of mathematical modelling*. Routledge.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics*, *102*, 9-28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Højgaard, T. (forthcoming). *Mathematical competencies in mathematics education: Past, present, and future*. Springer.

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring – Ideer og inspiration til udvikling af matematikundervisning i Danmark* [*Competencies and mathematical learning – ideas and inspiration for the development of mathematics teaching and learning in Denmark*]. The Ministry of Education.

Niss, M., Bruder, R., Planas, N., Turner, R., & Villa-Ochoa, J. A. (2016). Survey team on: Conceptualisation of the role of competencies, knowing and knowledge in mathematics education research. *ZDM*, *48*(5), 611-632. https://doi.org/10.1007/s11858-016-0799-3

OECD. (1999). *Measuring student knowledge and skills – A new framework for assessment*. OECD, Programme for International Student Assessment (PISA).

Schoenfeld, A. (1985). *Mathematical problem solving*. Academic Press.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In Grouws, D. A. (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). Macmillan.

Shavelson, R. J. (2010). On the measurement of competency. *Empirical Research in Vocational Education and Training*, *2*(1), 41-63. https://doi.org/10.1007/BF03546488

Stillman, G. A., Blum, W., & Biembengut, M. S. (2015). Cultural, social, cognitive and research influences on mathematical modelling education. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 1-32). Springer.

Turner, R., Blum, W., & Niss, M. (2015). Using competencies to explain mathematical item demand: A work in progress. In K. Stacey, & R. Turner (Eds.), *Assessing mathematical literacy* (pp. 85-115). Springer. https://doi.org/10.1007/978-3-319-10121-7_4

Undervisningsministeriet (2009). *Fælles Mål 2009 – Matematik* [Common Objectives 2009 – Mathematics]. Ministry of Education.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477. https://doi.org/10.2307/749877

Appendix A

Examples of tasks developed for and used in primary and secondary education with the explicit aim of facilitating the development of mathematical modelling competency (cf. the textbook series *Matematrix* and Jensen, 2007a, appendix C and E, my translation).

Invitations to develop... Edu. level	***Math. modelling comp. – inquiry-based projects (2-4 lessons/weeks)***	***Math. modelling comp. – desk-based work of short duration (within a lesson)***	***Mathematisation comp. – desk-based work of short duration (within a lesson)***
Lower primary	• Compare things found in the woods. • Create your own shop and visit your classmates' shops. • How many of you need to stand on top of each other to reach the ceiling? • How big is the school playground? • How many students can your school have?	• How high can you jump? • Who are tallest: the girls or the boys? • Draw a sketch of one of the rooms in your house. • How long does it take to read a book? • How long does it take to count to 1000? • How many students weigh the same as an elephant?	• How many more girls are there than boys in the drawing? • How long is the removal van when the boot is open? • What fraction of the balloons are either blue, red or yellow? • How many migratory journeys must a lapwing make to equal the distance flown by an Arctic tern?
Upper primary	• How many books are there in the school library? • How do you spend your time? • How much does it cost to have pets? • Draw a map. • What does a litre look like? • How much waste do you produce?	• How many rolls do you need to bring for a class breakfast? • How long will it take you to save up DKK 1000? • How much of the total area of the globe does Denmark fill? • What is the distance between your ears? • How big is your arm?	• There are 20 students in 4C at Brøkvild School. Of these, 9 are boys. Write the share of female students as a fraction. • Johan's father is 7 years older than Johan's mother. Their combined age is 83. How old are each of Johan's parents?
Lower secondary	• How much water do you use? • How much do I cost? • How can one navigate? • How many windmills should Denmark have? • What is the relation between one's income and the tax one pays? • Which means of transport is the best?	• Draw a floor plan of a 135 m^2 house. • Draw a graph showing how the temperature of a glass of water changes when you add ice cubes. • How much fabric is needed to make a tablecloth? • How many dices is there room for in a dice cup?	• Three children are to share DKK 450. Mark gets DKK 50 more than Eva, and Patricia gets twice as much as Eva. How much do they each get? • When you buy something, is it better to be given a percentage discount before or after VAT?
Upper secondary	• What is the best shape for a tin? • Is an escalator better than a normal staircase? • Can one lose weight by exercising? • How many molecules are there in a piece of chalk? • How does a bicycle computer work?	• At what angle of incline does a tower topple? • How far away is the horizon? • How far ahead must the road be clear for you to safely overtake? • What are the maximum dimensions of a board if one is to turn a corner?	• How is the overall tax burden affected by the rate of income tax and the rate of VAT? • A liqueur glass is cone-shaped. How far up the glass do you need to fill for it to be half-full?