Students' modelling processes when working with math trails

Processos de modelação de alunos envolvidos em trilhos matemáticos

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Abstract. On math trails, students can make direct connections between real objects and mathematical ideas. In these extracurricular learning arrangements, which are in the form of a rally, students visit places and objects in the city or around the school to solve mathematical tasks. The tasks comprise measuring or estimating relevant sizes and quantities and placing them in a respective mathematical model. One indicator of the usefulness of math trails as a form of learning is the extent to which they entail modelling processes. In the present qualitative study, two 11th-grade school classes at Oslo, divided into five groups of three students each (N_{groups}=10), were individually recorded on video while working with math trails. Each group's work was then analysed to determine observable modelling processes. The results for the groups in completing a math trail that involved circle calculation showed individual progressions between modelling phases while they worked on the tasks. The real objects were used in particular in various forms of data collection and validation. The article presents the study and reports on empirical findings on two groups of students' modelling processes on math trails.
Keywords: math trails; modelling; extracurricular learning; mathematising; Math & The City.

Resumo. Nos trilhos matemáticos, os alunos podem estabelecer conexões diretas entre objetos reais e ideias matemáticas. Nestes ambientes de aprendizagem fora da escola, sob a forma de percursos programados, são visitados locais e objetos existentes na cidade ou em redor da escola onde surgem tarefas matemáticas para resolver. As tarefas podem implicar medir ou estimar tamanhos e quantidades relevantes e integrá-los num modelo matemático apropriado. Um indicador da vantagem dos trilhos matemáticos como forma de aprendizagem é a possibilidade que oferecem de desenvolver processos de modelação matemática. Num estudo qualitativo, duas turmas de 11.º ano de Oslo, divididas em cinco grupos de três alunos cada (N_{grupos}=10), foram gravadas individualmente em vídeo no decorrer da sua atividade em trilhos matemáticos. O trabalho de cada grupo foi depois analisado em termos dos processos de modelação observáveis. Os resultados de dois grupos num trilho sobre medidas de círculos mostram progressos individuais entre sucessivas fases de

https://doi.org/10.48489/quadrante.23699
Received: February 2021 / Accepted: June 2021

modelação enquanto realizavam as tarefas. Os objetos reais foram utilizados, em particular, para várias formas de recolha e validação de dados. O artigo apresenta o estudo e relata os resultados empíricos relativos aos processos de modelação matemática de dois grupos no decurso de trilhos matemáticos.
Palavras-chave: trilhos matemáticos; modelação; aprendizagem extracurricular; matematização; Math & The City.

Introduction

Math trails provide extracurricular learning opportunities in mathematics, and they enable students to gain diverse experiences in applying mathematical knowledge in everyday contexts. In these learning arrangements, interesting mathematical places or objects in urban areas or in school surroundings are visited in the form of a rally, where mathematics tasks are worked on (Blane & Clark, 1984; Buchholtz, 2020a; 2020b; 2021; Cahyono & Ludwig, 2019; Gurjanow et al., 2020; Shoaf, Pollak, & Schneider, 2004). Depending on the task at hand, math trails can convey basic experiences of mathematical modelling. To promote mathematisation and validation skills that are central to mathematical modelling, the tasks must be appropriately application-oriented and not only providing routine problems (Buchholtz, 2017). For example, meaningful data collection can be integrated if the relevant quantities required for the mathematical solution are determined based on real objects through estimation or measurement activities and subsequently applied to a mathematical model (Greefrath, 2010).

Mathematics trails, which have existed since the 1980s, recently have been rediscovered as an extracurricular learning opportunity in mathematics. In the past, they were mostly created and used informally in leisure-time educational offers. Recently, however, they have been used to supplement mathematics lessons in the classroom. The emergence and use of mobile devices such as smartphones and tablets have enabled digital support for math trails, which also provide added value for presenting and working on the tasks (Buchholtz, 2021; Cahyono & Ludwig, 2019; Gurjanow et al., 2020). Examples are when multiple representations of the task content are provided or additional information about the real object is given in photos.

From an empirical perspective, although task analyses may provide insights into the specific modelling requirements of tasks, little is known about what being outside and working with real objects can contribute to the learning of modelling. There is still a lack of evidence on the kinds of modelling activities mathematics trails can provide when students work on the tasks. Hence, empirical research on the learning outcomes of mathematics trails is still in its infancy (Cahyono, 2018; Ludwig & Jablonski, 2019; Zender, 2019). In the present qualitative explorative study *Math & The City*, two 11th grade classes at a school in

Oslo/Norway were divided into five groups of three students each (N_{groups}=10). All groups were videorecorded while performing digitally supported math trails (Buchholtz, 2020b). Their processes in completing the tasks and their interactions with real objects and with each other were analysed from the theoretical perspective of mathematical modelling.

The aim of this study was to analyse students' task-specific modelling processes and strategies in working with a math trail involving circle calculation. Their interactions with objects connected to different tasks on the trail were observed, as well as their mathematisation, interpretation, and validation strategies in estimating and taking measurements. Exploratory qualitative research methods were used to analyse the students' modelling processes and collect evidence for the potential of math trails to promote modelling activities.

Theoretical Background

Mathematising activities in contextual situations as a key to modelling

Mathematisation processes play a special role in task design for math trails. The term "mathematising" originally goes back to Freudenthal (1968, 1973) and his engagement (together with Treffers) in the Dutch curriculum reform project Wiskobas in the late 1960s and early 1970s, which is often referred to as the origin of the Realistic Mathematics Education (RME) approach in the Netherlands. According to Freudenthal (1968), mathematics should not be learned as a closed system but as an activity of mathematising "reality" and, if possible, even by mathematising mathematics. Here, "mathematising" refers to mathematical structuring in the sense of a transition from the lifeworld to the world of symbols. In later publications, Freudenthal accepted Treffers' (1987) distinction between horizontal and vertical mathematising, in which the former means making a lifeworld problem field accessible to mathematical treatment and continuously schematising it, and the latter means mathematical processing as an increasing process of abstraction. In its original meaning, the term refers to basic mathematical activities, such as counting, structuring and comparing quantities or illustrating basic arithmetical operations, such as adding, multiplying and dividing, thereby referring to almost all mathematical activities (cf. examples given by Freudenthal, 1991, p. 42-44). In situated work conducted with physical representatives of typically idealised mathematical objects, measuring and determining quantities, as well as the structural comprehension of mathematical relationships between physical objects (e.g., order or geometrical pattern or size comparisons), is therefore understood as mathematising. Freudenthal's understanding of mathematising is broader than that in today's discussion on modelling. The horizontal meaning of the term is now used specifically to refer to processes of description and translation from reality into mathematics (i.e., analogously, "interpreting" as back-translation) (Blum & Leiß, 2007; Niss, 2010). This usage does not necessarily include data collection, which is sometimes considered more an activity of *structuring*. However, both mathematising and interpreting are broadly understood as central components of mathematical modelling (cf. e.g. Greefrath & Vorhölter, 2016; Kaiser, 2007; Niss, Blum & Galbraith, 2007), and they are regarded as important in students' learning of mathematics. In the literature, modelling is usually ideally described as going through a circular process in which essential activities or subcompetencies play a role. This process consists of simplifying problems, making appropriate assumptions, translating a real problem into mathematics (mathematising), working mathematically, relating a mathematical result to the real situation, and interpreting and validating solutions (Blum & Leiß, 2007). Processes such as structuring and simplifying problems using real objects, calculating with real quantities, and object-related validations of mathematical results are therefore central activities in extracurricular learning with math trails.

Contextualisation of tasks in math trails

In the math trail design used in the present study, the tasks followed experience-based criteria, such as relating to content previously treated in class, having a realistic problem orientation, encouraging students to mathematise using determined sizes and quantities, and including diverse mathematical concepts and ideas (Buchholtz, 2017). The process of mathematising within the tasks, however, occurs in comparatively small, guided steps, and the developed mathematical models were not complex, as the students working on the trail were inexperienced modellers. However, these constraints beg the question of whether the tasks should be regarded as "classic" modelling tasks or as effective mathematical outdoor activities. Figure 1 shows an example of a task that was part of a math trail in the *Math & The City* study, which provided the foundation for the analysis of students' modelling activities described in this article.

The well-known Peacock Fountain in Oslo.
When the fountain is turned on and the pool is filled in the spring, the city council must find out how much water the pool holds.

a) For this purpose, the perimeter of the pool must be determined.
b) Next, find the area of the pool.
c) How much water can the pool hold?

Provide your answer in m^3.

Figure 1. Fountain task (Buchholtz & Singstad, 2021)

In the fountain task, the volume of water must be calculated. The task is embedded in a real-world context, in which the pool must be filled with water every spring because the winters in Oslo are characterised by extremely low temperatures. The city council therefore needs to know how much water is needed to fill the pool. In this task, the students must consider how to determine the volume of water in the cylindrical fountain. One important difficulty is that the diameter of the basin (7.6 m) is not accessible and therefore cannot be measured. Therefore, a suitable mathematisation must be determined based on the quantities that can be measured (perimeter = 24 m). For example, the students could choose the number of footsteps as a non-standard unit of length by walking around the pool, counting the number of steps taken, and then multiplying it by the length of one step. The students must note that the stones are measured at the inner edge of the basin. Otherwise, the results will vary considerably. Based on the corresponding water level of the pool (0.36 m), the water volume (16.5 m^3) could then be approximated.

Previous studies in the literature have paid little attention to real-world representational contexts in the classification of modelling-related tasks. Ludwig and Jablonski (2019) mentioned that students on math trails "have to think about which data they have to measure. This is really the difference to modelling tasks in the Classroom [sic!]" (p. 907). However, embedding a task in a real-world context is possible to varying degrees. For example, Hagena (2019) compared various tasks. In one task, the photo of a real brick tower was shown, and the task involved determining the number of bricks needed to build the tower in the image. However, there are differences when the brick tower would be actually visited on a mathematics trail and the students would determine on site the number of bricks used by both calculating it and by making various assumptions. In contrast to the photo, the visual-haptic presence of the object and its actual size would allow for a broader range of approaches to obtaining exact results and an easier interpretation of the mathematical solution, especially in relation to the students' data collection. In validating solutions, the direct comparison between calculations and real objects enables an empirical verifiability that is not possible in the classroom. Therefore, tasks on math trails should always include the students' collection of real data on objects, which ensures that the required mathematising is contextualised.

To demonstrate the importance of realistic contexts when modelling in math trails, Buchholtz (2020a) developed an idealised model of real-world context-based modelling processes that can be observed on math trails (Figure 2). This model was used to identify the central activities of students in working on the math trail in the present study.

The representation resembles a modelling cycle (Blum & Leiß, 2007; Greefrath & Vorhölter, 2016; Kaiser & Stender, 2013) or the *nodes model* of modelling activities described by Doerr and Pratt (2008, p. 264), which includes the phases of mathematisation, mathematical work, interpretation and validation. On math trails, the transition between reality and mathematics, or back transition in the context of the objects, plays an important role. Therefore, the model includes spheres of contextualisation, which are the steps in which the individual modelling processes are closely linked to real objects. For example, in math trails, data collection and the localisation of the quantities to be collected from the real object play key roles.

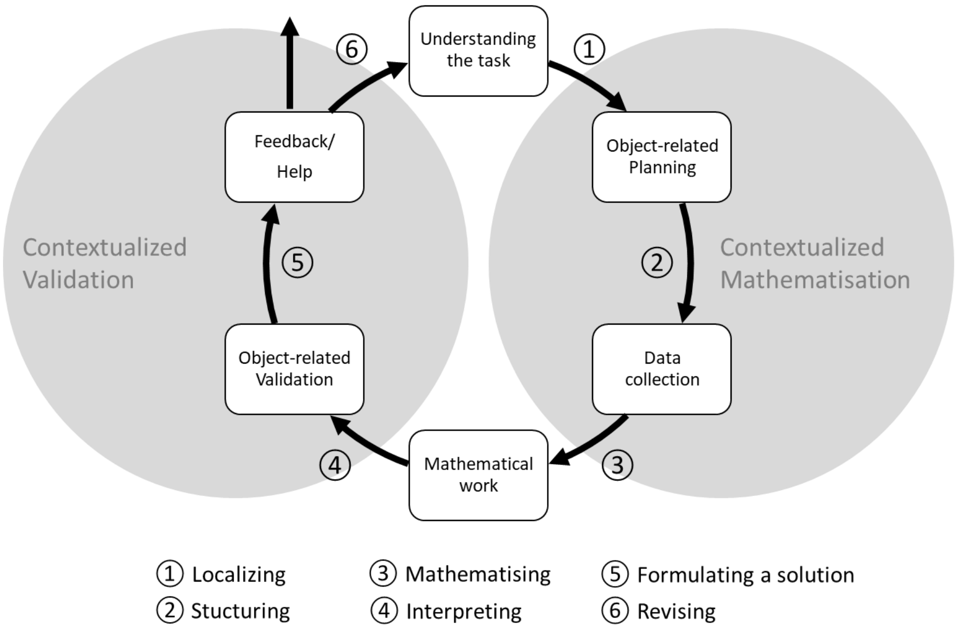

Figure 2. Modelling processes on math trails (Buchholtz, 2020a)

Research question

Based on the theoretical background, this paper addresses the following research question: What kinds of contextualised processes (e.g., mathematisation, interpretation, and validation) do 11th grade students show on a math trail when they encounter and interact with objects in the task?

Method

Data collection

In conducting the math trails with students, each group of students was equipped with a tape measure, something to write on, and a mobile device (iPad) that displayed the tasks and guided the students on the trail based on geo-location coordinates. The students submitted all solutions to the task via an app[1]. Both groups of students were also equipped with an action camera belt that recorded their progress on their math trails. The angle of the camera was directed to the floor so that it recorded the working processes of the students on the iPads and—most importantly—with the real objects in the task. The video recordings also ensured that the personal rights of third persons were not harmed when the students moved through the city (Figure 3).

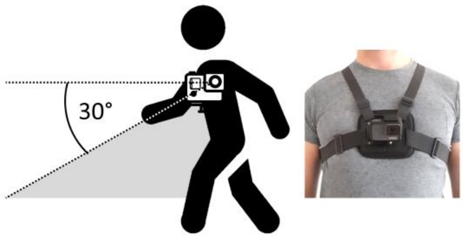

Figure 3. Data collection method

The students were responsible for recording their processes at individual stations along the math trail, which ensured that the data collection was minimally invasive. In November and December 2019, the study was conducted with two 11th grade classes[2] from Oslo and their mathematics teachers. In each of the two classes, five groups of three students were equipped with cameras and iPads. One trail, which was focused on circle calculations, was based on five tasks (including the fountain task), and the other trail, which was focused on linear functions, was based on four tasks. The data were collected from (N_{videos}=45) videos between 8–20 minutes in length. The video data were analysed using the software InterAct, and the students' solutions were saved in the app.

Data analysis

Methods used in qualitative explorative research were applied to evaluate the math trails. The qualitative method of the narrative walk-in-real-time interview (*méthode des itinéraires*) was originally developed to collect and describe subjective views of pedestrians in order to draw conclusions about urban planning (Miaux et al., 2010; Petiteau & Pasquier, 2001). Central to this method are specific city walks, whereby the researcher takes a more or less (depending on the study) passive role in being guided by a participant. The researcher interviews and records the participant, followed by a photographer who videorecords each change in direction or emotion. This method was adapted for application in math trails to gather information about modelling activities and subjective views about the learning contexts of the tasks (Buchholtz & Singstad, 2021). The data collected from the video recordings are then analysed using qualitative content analysis (Mayring, 2014).

In the present study, a combined deductive and inductive approach was chosen for the analysis of the video material. In the first deductive phase, we started with a predefined set of categories, which were formed with the help of relevant theoretical literature on different modelling processes (Blum & Leiß, 2007; Freudenthal, 1991). The theoretical categories of *understanding, structuring, mathematising, calculating, interpreting* and *validating* were included in the analysis (Table 1). Activities in goal-oriented data collection, such as counting, estimation and measurement, were assigned to *mathematising* in Freudenthal's sense (Greefrath, 2010; Kuch, 2018). The data material was carefully coded by continuously time-stamping individual sequences in the videos in which corresponding modelling activities specified in the categories were visible based on various indicators. Even if erroneous approaches were used in the process, the underlying activity was initially considered a modelling process to gain as comprehensive an impression of the activities as possible. An advantage of using predefined categories was that the video material became more manageable and the complexity could be reduced. The coded processes had different durations, but both the frequency of the sequences and the length of the sequences were used to address the research question in this study.

Table 1. Overview of deductive categories

Theoretical categories (modelling activity)	Definition	Indicators (examples)
Understanding (Forstår)	Students spend time reading and understanding the task.	Looking at the iPad and reading. Discussing the task. "What shall we do?"; "Have you understood what we are going to do?"
Structuring (Strukturere)	Simplifying. Students recognise the relevant physical object and sizes and create a mental (real) model.	Locating the place of the object. Using the object in the planning. Delegation of work.
Mathematising (Matematisere)	Students' attempts to express the problem mathematically – transferring the real model into mathematics.	Collecting the information needed. Estimating, measuring, counting and putting the relevant quantities into a mathematically meaningful relationship.
Calculation (Begregne)	Students calculate the problem, which is now mathematically, it is solved and calculated by using mathematical knowledge.	Using measurements to calculate a solution. Using the calculator on mobile phones.
Interpretation (Interpretere)	Students interpret the mathematical solution and translate it to the real world.	Interpreting and explaining what the different numbers they arrived at mean.
Validation (Validere)	Students find out if their solution agrees with reality and whether the answer gives meaning.	"No, it cannot be right!" and looking at the object again. Getting results in the app. Looking at the answer in the app and comparing it. Troubleshooting.

In phase two of the analysis, the videos were coded inductively. In this phase, additional categories were formed based on specific events that emerged from the data. The aim was to obtain further unexpected findings that could be relevant in addressing the research question. First, noteworthy activities of the students in working on the math trails that could not be described by the deductively obtained categories were identified. If the same or similar activities were then observed in several videos, these activities were included in the coding scheme as an additional category of analysis. All videos were examined to determine whether they included data in these categories, and their occurrences were given frequency codes (i.e., singular time stamps). These categories included the use of digital media and specific group interactions. As in the first phase, the inductive categories that were identified in the videos were added to the InterAct software. The inductive codes assigned were *Googling, Individual Work, Imitating, Misunderstanding* and *Other* (Table 2).

The categories and the frequency of both types of assigned codes were interpreted in the analysis of the group activities. This sequence-oriented approach has previously been applied in the context of video-based research on modelling and problem solving in previous studies (e.g. Modelling Activity Diagrams: Ärlebäck & Albarracín, 2019). As a tool of analysis, it has advantages over the circuit model or arrow diagrams. The resulting diagrams yielded better impressions of individual modelling processes (Borromeo Ferri, 2007), in which, for example, phases can overlap or repeat.

Table 2. Overview of inductive categories

Emerging categories (inductive)	Definition	Indicators (examples)
Googling (Google)	Students decide how to perform various tasks.	The camera captures the students' Googling. Students say they should Google to find something out or find a formula.
Individual work (Individuelt arbeid)	A student takes on the work, while the other students do not work despite the fact that it is the task fosters cooperative work.	A student takes the iPad and sits down to solve the task. A student stays in the background, solves the task on their own, and comes back to the other students with the answer.
Imitation (Hermer)	See what the other student groups do, and do the same.	The student group is seeing onto another group of students and talking about doing the same. A student captures what another group does and tells the others that they have an idea of how the task can be solved.
Misconception (Misoppfatning)	Students say something that is erroneous, or it is obvious that they have a misconception related to the situation.	"Okay, so the formula for the area of a circle is pi squared times the radius. Then we have to take 3.14 times 2, which is 6.28 times the radius. " "Yes, 1 metre, ... it's 60 cm." "Diameter is the same as perimeter."
Other (Annet)	Miscellaneous occurrences	Students begin to do other things as a result of lost motivation or misunderstanding of the task. Talking about other things. Playing football. Checking social media. Putting on music. Complaining that it's boring.

Results

For the purposes of this paper, the modelling processes of two groups in performing the fountain task (Figure 1) will be described in detail. Figures 4 and 5 show diagrams with the overlapping codes created for each group of students for their task processes in InterAct. In the upper parts of the diagrams, different colours indicate the modelling processes during the temporal course; the lower parts show time marks for the assigned inductive codes. The groups differed in their achievements: group 2 encountered many difficulties during the task-solving process, and group 3 accomplished the task smoothly. In addition to the diagram, parts of the video sequences are summarised and interpreted in subsequent tables (Tables 3 and 4) that show how the data were coded. Furthermore, the interaction of the students allowed for an interpretation of the solution process and detailed descriptions of the modelling activities that took place. Tables 3 and 4 contain descriptions of the data, quotations of the students' dialogue, and interpretations of the coding.

Modelling processes in group 2

Group 2 consisted of three girls (S4, S5, and S6) who expressed that they needed much support in mathematics and that they neither did particularly well nor were interested in mathematics. S4 had repeated the 11th grade after she dropped out of the subject in the previous year. The students in group 2 showed positive attitudes, but they were less motivated and spoke less compared with the other groups. The average correct response rate of group 2 was 64%. Group 2 completed the task in approximately 15 minutes, which was the longest of all the groups. The group had as many as 20 misconceptions in their work on this task, and they Googled 14 times. Moreover, group 2 may have been successful in the subtasks at the end because they imitated another group at the beginning of the work. At the end of the task processes, the group was highly engaged in mathematical calculations (Figure 4). However, their misconceptions about using different measurement units led to increasing confusion in the group.

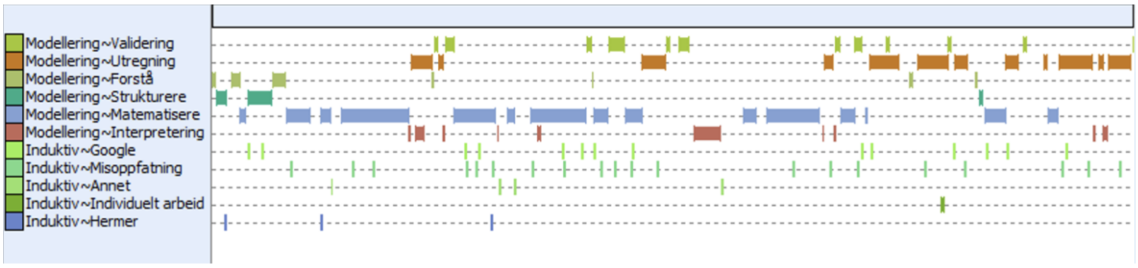

Figure 4. Modelling process of group 2 on the fountain task (note: the codes are provided in Tables 1 and 2)

The coded sequences (Table 3) show that group 2 used different approaches to mathematisation, and it was creative in determining relevant variables. For example, the first approach of direct measurement was immediately discarded when the approach of a different group was observed. Group 2 decided to fall back on the use of embodied measurement units, such as steps of one metre to determine the perimeter in task A. The imitation of the other group showed these students how to proceed, but the group members also developed respective approaches by themselves. However, they did not remember the correct formula for the perimeter, so a mistake made by S5 confused the group. This group lacked the mathematical knowledge required to correctly convert the formula for the perimeter to determine the length of the radius of the fountain in task B. When they did not find a correct solution even by Googling, the group decided to obtain further data on the radius using the tape measure, but the result was imprecise because they measured the radius in the water. Based on countable entities in the real objects (stones), the group therefore finally decided to estimate the radius. In task C, they made another mistake by mixing up the measurement units. It was observed that group 2 had a quick grasp of the meaningfulness of data collection. However, the group's inability to put the collected variables into a mathematical model hindered their successful mathematisation.

Table 3. Description of Modelling Activities of group 2

Modelling process	**Codes and interpretation**
Reading the task. Thinking about it and looking at each other. S5: "Wasn't that what we did in the last task? What do we use? Was it diameter times pi?".	Understanding: connecting the task immediately to the previous task. Trying to remember the formula.
Finds out that it gets difficult to measure with a measuring tape all the way over the fountain to find the diameter. Group tries it.	Mathematising: making an attempt to measure despite the fact that they realise that this doesn't work.
The measuring tape is too short, and they have to use another method. They see that the students in another group walk around the fountain and count the number of steps. They want to do the same.	Imitation: see what the other group is doing and choose to do the same though they do not know if it is the right way.
S4 tells the others that they must measure up one metre on the measuring tape and then try to go with steps that are exactly one metre. Walking around the fountain. Gets 24 steps.	Mathematising: make a new attempt to measure the perimeter with individual units of measurement.
Checking the task on the iPad. S5: "Shouldn't we multiply this by 3.14 now?" Became quickly aware that it was the perimeter the task A asked for. Enter 24 m and get the right result.	Misconception and Interpretation: Follow up on the idea of S5. But relate their results to the fountain. Try to remember the formula for perimeter, but in the end stick to the data they collected.
Do not know how task B can be solved. Google the formula and find that they need the radius.	Googling and Mathematising: Searching for anything to get the work on at all. Identifying the right formula.
S5: "Okay, so the perimeter is 24, then the diameter must be 12, and radius is 6". Use the calculator to find pi times r squared.	Misconception and calculation: They have not understood the relationship between perimeter and diameter.
Result seems too big. S4: "Is the radius 6 metre?". Looking at the fountain. "It can't be right."	Interpretation and validation. They see that the radius of the fountain must be less than 6 m. They get stuck.
Do not understand the formula. Trying to measure the radius with tape again. It's too far over the fountain, and the measuring tape gets wet. S6 says that the stones at the bottom of the fountain look like having a metre length.	Mathematising: first approach of measuring directly fails, but attempt number two with smart counting works better.

Asks how many such stones there are from the edge of the fountain basin to the centre. Comes to the conclusion that the radius is 3.5 metres.	Individual units of measurement are used here.
They want to calculate pi times 3 squared. Guess the answer is 9.42. Say they have no idea how to calculate pi times r squared. Guess the answer, but the app shows it's wrong (correct answer is shown).	Calculation and Misconception: It looks like they calculate pi times 3, but do not know how they should square 3.
Task c) How much water does the pool hold then? Get help from teacher to understand that the difference between area and volume is the height. Measure therefore the height with measuring tape. Read the number of centimetres. Use calculator to find the area (which they got in m^2). They multiply times the height (which they have found in cm). Do not understand the result.	Interpretation: Need help to interpret what they are doing. Then try to figure out themselves but encounter one barrier when mixing units of measure. Misconception: Lack of validation.

Modelling processes in group 3

Group 3 consisted of three boys (S7, S8, and S9), all of whom were considered by the teacher to be strong in mathematics. Nevertheless, the group only managed to achieve an overall average solution frequency of 73%. Group 3 solved the fountain task in 11 minutes. Compared with the other groups, the group needed little time to understand the task and to structure the work. Furthermore, they validated their results frequently. Figure 5 shows that the modelling process in group 3 ran almost idealtypically. That is, the processes of understanding, structuring, mathematising, calculation, interpretation, and validation were observed in a following order.

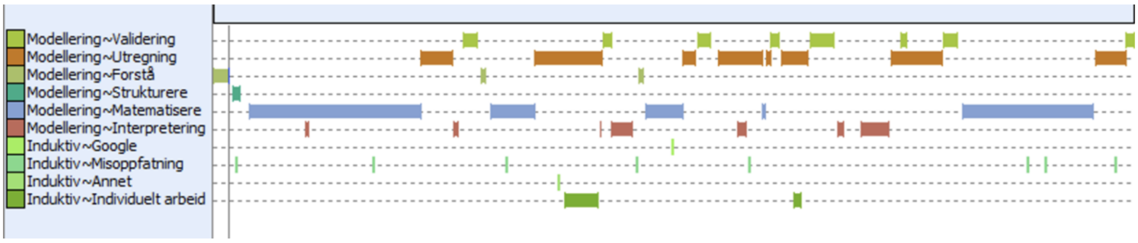

Figure 5. Modelling process of group 3 on the fountain task (note: the codes are shown in Tables 1 and 2)

The analysis of the coded sequences (Table 4) revealed that the group 3 worked in a relatively goal-oriented and planned manner. They worked with individual units of length that they measured on the real object. The students in group 3 helped each other and explained the differences between the relevant mathematical concepts (i.e., perimeter and area), but in the course of the task process, it was observed that S7 led in the individual work and the other two students did not understand the approach as quickly. However, at the end of the task, the students again worked as a group because further data had to be collected (i.e., height of the water level). Rearranging the perimeter formula to determine the radius and then the area caused problems in the group, but they were able to overcome the difficulties themselves with minimal help from the teacher. They also developed object-based validation strategies that helped the group identify errors in units.

Table 4. Description of Modelling Activities of group 3

Modelling process	**Codes and interpretation**
Group reads the task a) text. The students remind each other that perimeter is "that which is around, not inside."	Understanding the task
Thinking of a way to solve this. S7: "We cannot use the measuring tape because it is too short. How long is the measuring tape?"	Structuring: Creates a plan but checks before execution whether it is wise to implement the plan.
S7 Looks at the rocks at the bottom of the fountain and discusses if they are alike big. Then looking at one of the rocks outside the fountain, which go around the whole fountain. They measure one stone and count how many stones there are around. Each stone is 125 cm.	Mathematising: Object-related planning and data collection; group follows the idea of smart counting and measuring with individual units of measurement provided by the object.
S8 calculates how much 125 cm times 20 stones are. Almost right, the app shows only a 1 m deviation.	Calculation and interpretation
Checking out task b) and relating it to the fountain. S9: "What is the area again, it is the around, right?" S7: "No, it's inside!"	Understanding and task validation: Group collaboration helps to quickly validate the correct understanding of the task.
To find the area of the fountain, S7 uses the perimeter and divides with pi to find the diameter. S8 and S9 do not remember these formulas.	Calculation: This goes a little too fast for S8 and S9. They seem to lose motivation.
S7 remembers the formula and finds the radius by dividing the diameter by two. Then he says that one should only multiply the radius by pi to get the area.	Calculation and individual work with misconception
They Google the formula.	Googling and Validating of the approach
Get the number 12 by multiplying the radius by pi. S9: "But it's probably more than 12 square metres here ?!" They answer "12", but it's wrong in the app.	Interpretation and object-related validation: They check their result by comparing it with the fountain and individual ideas about the actual size. But do not do something with what they think is wrong.
Teacher helps: "It is not r times pi. It is pi times r squared". The group calculates on the calculator now and gets 45. That gives the correct answer.	Validation: Get help to correct errors.
Task c) How much water can the fountain hold? S8 remembers the formula "Base times height". The students quickly agree that the base is the area that they just found. The height is about 38 cm. S8 measures the water height.	Mathematising: Recalling the formula for the volume and agreeing on how to bring the quantities together, data collection
Enter 45 times 38 on the calculator, which gives the answer 1710, but they are aware of the units of measurement, so they write 17,1 cubic metres as reply and get it correct.	Calculation and interpretation: they remember to convert from cm to m

Discussion

The video recordings showed the individual groups of students solving the required tasks during the math trail. Different modelling processes were observed. The results showed that, similar to other modelling activities, math trails with corresponding tasks can, based on these findings, also be described in terms of the modelling processes involved. As the analysis of the videos revealed, structuring, mathematisation, interpretation, and validation activities played a particularly important role in the students' solution processes. Transitions between reality and mathematics had to be realised, which involved individual content-related ideas and prior mathematical knowledge, such as the relations and differences between perimeter, area, and diameter and the corresponding formulas used to determine them. However, when the students lacked mathematical knowledge, misconceptions hindered them from achieving adequate mathematisations and correct solutions. An interesting finding was that when students were stuck in solving a task, they imitated the solution approaches used by other groups, or they quickly tried to Google a solution. This finding on the one hand indicated the helplessness of the students, but on the other hand it was also observed that independent thought and creativity was developed through the help of others.

In contrast to regular modelling tasks in class, the contextualisation when mathematising and validating via the real objects seems to play a special role for math trail tasks. To solve the tasks, the students had to measure, scale, count, or estimate quantities and then place them in a correct mathematical relation or reconstruct or calculate relevant but inaccessible quantities from measured quantities – that is, actual mathematising.

When understanding the tasks, the students localised the relevant quantities in the real objects, which was evident in the videos through corresponding gestures and the viewing directions of the students. Here, the extended contextualisation through the real objects becomes relevant for the first time, as shown in Figure 1 by the transition to grey spheres. To plan a solution approach and determine specific required sizes, the determined quantities had to be related to the corresponding real object (S9's question "What is the area again, it is the around, right?" in group 3, see Table 4). Context-related assumptions or simplifications were made. For example, the basin of the fountain was cylindrical, and the technical fittings of the water fountain in the basin did not have to be considered. Such localisation and object-related planning is comparable to the creation of a real model of a situation in classical modelling tasks in the classroom. However, the approach of mathematising first requires a structuring of the lifeworld context, in which mathematical ways of thinking and working, as well as known heuristics, play a role, without already working purely mathematically here (Freudenthal, 1991). However, the results showed that this structuring was observed in only a few situations, such as when the groups either tried an initial approach directly (group 2) or created a plan (group 3). This was followed by targeted data collection through measurement activities or estimation. Based on the accessibility and size of the real object and the available tools, suitable ways of data acquisition have to be found for this purpose, partly using the conditions of the real object. However, as previously mentioned, in many theoretical modelling cycles, data collection is not identified as an explicit step (Blum & Leiß, 2007; Greefrath & Vorhölter, 2016; Kaiser & Stender, 2013), so its theoretical relation to the understanding of mathematising is not yet clear. It appears to have been marginalised for two reasons: first, when working with classical modelling tasks in the classroom, it is unresolved whether data collection belongs theoretically in the realm of simplifying, mathematising or mathematical work. Second, it seems to be negligible insofar as it consists only of the pure information acquisition (given quantities) from the task formulation or an additional illustration (e.g., a photo of a hot air balloon). However, regarding math trails, it seems reasonable to theoretically identify data collection as an explicit step in contextualised mathematisation because it constitutes the core of the contextualisation extended by real objects. When the relevant quantities have been determined, they must be put into a suitable mathematical context. For this purpose, content-related ideas are activated, for example formulas for calculating quantities are used, such as those observed in group 3 in the present study. Hence, the transition to the mathematical model via mathematisation takes place.

During the math trails, it was observed in the video recordings that the students mentally detached themselves from the real-life context, such as by performing calculations on paper or a smartphone. This step in mathematical work, which is understood as symbolic operation, therefore seemed to be a purely cognitive activity outside the sphere of contextualisation. The mathematical results were subsequently interpreted with respect to the task and, if necessary, by re-localisation, thus switching back to the lifeworld contextualisation of the real object. The groups directly checked their mathematical results against the real objects. The findings revealed indications of several direct validation processes (Czocher, 2018), such as S9's (group 3) comparison with known benchmarks: "But it's probably more than 12 square metres here?!". In looking at the fountain, S4 in group 2 asked: "Is the radius 6 metres?" – "It can't be right". By having the real objects as physical entities available, as a control strategy, the results could be validated directly on the object (Borromeo Ferri, 2006; Czocher, 2018).

Limitations

The results of the study were based on the analysis of the video recordings of individual groups of students. Therefore, the results must be interpreted with caution, and they cannot be generalised. The evaluation of the data collected from the video recordings was based on a qualitative coding procedure, which entailed a necessary coarsening of the evaluation categories so that several videos could be economically compared with each other to address the research question. Because this study did not focus on cognitive processes, a further replicative study could conduct a deeper analysis to determine the factors that influenced the individual groups in their solution processes and the effects of outside help (e.g., imitation, the teacher, or Google) on their solution process. Because the prior knowledge of the students was not controlled, and other performances were not measured, no conclusions could be drawn about the effects of math trails on the students' mathematical abilities beyond the observation and identification of situational modelling activities.

Closing remarks

If math trails provide real-life contexts for modelling activities, they could be a useful supplement to mathematics lessons in the classroom, giving students an opportunity to apply mathematics in their everyday lives. The students' modelling processes examined in the present study were embedded in an extended contextualisation that was not comparable to classical modelling tasks in the classroom. In the present study, contextual mathematisation and validation processes involved real objects, which were the basis of the data collection and the interpretation or validation of mathematical solutions. However, empirical research on the effects of math trails on learning modelling and mathematics is only beginning. Nevertheless, the results of this study could provide initial qualitative empirical findings regarding modelling processes used in math trails, which should be extended by future systematic research.

Acknowledgements

The project Math & The City is funded by DIKU, the Norwegian Agency for International Cooperation and Quality Enhancement in Higher Education.

I thank Juliane Singstad for the data evaluation.

Notes

[1] Details about the task presentation, the use of the mobile device and the app by the students are provided in Buchholtz (2020b).
[2] Students in the 11th grade in Norway are about 16 years old.

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