Technology and mathematical modelling: addressing challenges, opening doors

Tecnologia e modelação matemática: enfrentando desafios, abrindo portas

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Abstract: In terms of achieving educational goals, technology impacts on the nature of mathematical accomplishment with respect to both scope and purpose. We review the use of technology, actual and potential, within mathematical modelling viewed as real-world problem solving. We consider its role within the total modelling process, as well as its manner of use within individual problem contexts, illustrating ways in which inappropriate uses of technology create problems within modelling activity, as well as how discerning use can increase the power and accessibility of models to new audiences. We then demonstrate how technology provides access to models unavailable to those equipped only with hand methods of solution. Here non-linearity and simultaneity among model relationships means that model equations need to be first developed, parameterised, and then solved by simulation. Methods provided by System Dynamics are illustrated by considering the problem of providing potable water for a population expanding into a warmer environment, with limited water reserves.
Keywords: modelling; real-world; simulation; system dynamics; technology.

Resumo: No que se refere a atingir objetivos educacionais, a tecnologia tem impacto na natureza do desempenho matemático, tanto no seu alcance como no seu propósito. Fazemos uma revisão da utilização da tecnologia, real e potencial, no âmbito da modelação matemática, entendida como resolução de problemas do mundo real. Consideramos o seu papel ao longo do processo completo de modelação, bem como a sua forma de utilização no contexto de problemas concretos, ilustrando situações em que a utilização inadequada da tecnologia provoca perturbações na atividade de modelação, bem como outras

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em que o seu uso criterioso pode aumentar o poder e a acessibilidade dos modelos para novos públicos. Em seguida, demonstramos como a tecnologia permite o acesso a modelos que ficariam indisponíveis se apenas fossem usados métodos manuais de resolução. Neste caso, a não linearidade e a simultaneidade que têm lugar entre as relações do modelo indicam que as equações do modelo têm de ser primeiro desenvolvidas, parametrizadas e, em seguida, resolvidas por simulação. Os métodos fornecidos pela Teoria de Sistemas Dinâmicos são assim ilustrados, considerando o problema de fornecer água potável a uma população que cresce num ambiente que se torna mais quente, com reservas de água limitadas.
Palavras-chave: modelação; mundo real; simulação; dinâmica do sistema; tecnologia.

Introduction

This is essentially an illustrated theoretical paper with two main purposes. Firstly, to selectively overview aspects of technology as an agent within mathematics education, with specific reference to its contribution, actual and potential, in supporting mathematical modelling as real-world problem-solving: technology can both help and hinder the development and practice of modelling expertise. Secondly, to illustrate how technology can open the door to the development of models for problems which are otherwise inaccessible when only hand methods of solution are available.

The modes of technology use within mathematics education are so pervasive, and so varied that it seems necessary to circumscribe the boundaries of what a paper such as this can reasonably aim to cover. This is managed by focusing on technology as a support for mathematical modelling as real-world problem solving.

Here expectations are formally impacted by different meanings attributed to mathematical modelling in terms of its purpose within education, and it is instructive to compare different emphases with the priorities for mathematics education contained in national statements. For example, the U.S. Common Core State Standards Initiative (National Governors Association, 2010) describes a mathematically proficient student in these terms (p. 7):

> Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.
> Mathematically proficient students are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using mathematical tools. They can analyze those relationships mathematically to draw conclusions, and routinely interpret mathematical results in the context of the situation, and reflect on whether the results make sense…

Similar sentiments are contained in other national statements, for example the Australian Curriculum Assessment and Reporting Authority (2017), and the OECD (2021) *Description of mathematical literacy for 2021*. All mention workplace, citizenship, and personal competence as achievement targets in terms of applying mathematics to real problems. The common feature is to enable students to learn abilities to address problems in the world outside the classroom, with benefits that extend beyond their years at school. It is this view of modelling that is promoted in this paper, for there is no other place within the mathematics curriculum where such abilities can be developed.

This recognition of mathematical modelling as content in its own right, contrasts with its use as a vehicle to serve other curricular purposes, e.g., as an approach to learning content that can be approached by other means (Julie, 2002). Several varieties of modelling as vehicle are described in the literature (e.g., Kaiser & Sriraman, 2006). The distinguishing characteristic between the two fundamentally different genres (content and vehicle) in terms of educational practice, lies in where the ultimate authority is located. For modelling as vehicle, this lies with perceived curriculum and classroom requirements, prescribed either officially or by teacher choice. When modelling is tasked in this way to serve other priorities, its integrity as a problem-solving process stands to be compromised. For modelling as content, authority resides with the conduct of the modelling process, in terms of its relevance and quality for solving problems in terms of their real-world implications. Classroom practices will sometimes need to change as a consequence. For purposes of this paper "modelling as content", and "modelling as real-world problem solving" are used interchangeably.

The modelling process

Mathematical modelling as real-world problem solving is a structured process, which needs to be understood and practised before it can be applied confidently in new situations. While the following representation will be familiar to many readers, for completeness the common modelling cycle is summarised and illustrated below.

Different representations are in use, and diagrammatic depictions do not indicate the actual sequentially ordered itineraries (modelling routes) followed by individual modellers. Rather, such diagrams depict analytic reconstructions of the ordered components that are identifiably present in the complete solution to any modelling problem and its subsequent reporting (Niss & Blum, 2020).

The representation here (Figure 1) is adapted from Stillman et al. (2007). Entries A-G denote the phases in the modelling process, with thicker clockwise arrows indicating transitions between phases as the process proceeds. Descriptors 1-7 indicate the type of mental activity modellers undertake as they engage with respective solution sub-processes. The double-headed arrows indicate forwards and/or backwards looking reflective activity between phases. Transitions may occur in a non-sequential manner between *any* components across the cycle, but additional arrows have been omitted for the sake of diagrammatic clarity.

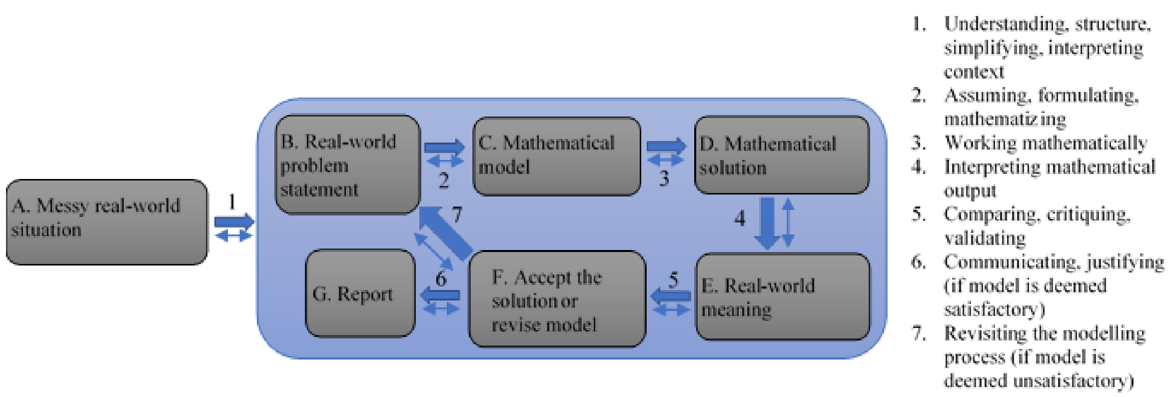

Figure 1. The process of mathematical modelling (adapted from Stillman et al., 2007)

Technology and mathematical modelling

In this paper we are concerned specifically with technology as it relates to mathematical modelling as content, and this involves the application of selective judgment. In particular, it makes sense to focus on aspects of technology whose use in mathematics education can be linked directly to the practice of modelling.

The survey paper by Clark-Wilson, Robutti, and Thomas (2020) reviewed ways in which different uses and aspects of technology impacted educational practice. Initially introduced as a support for new ways of doing and representing mathematics e.g., the use of *Cabri Geometry* software, it subsequently developed a social impact as a means of changing cultural characteristics of classrooms and methods of learning. It did this through facilitating new ways of generating and sharing information, supporting collaborative activity, and sharing materials (e.g., Goos et al., 2003). Insights from such sources feed directly into enhancing the performance of collaborative teams such as those which enter the International Mathematical Modeling Challenge (Garfunkel et. al., 2021).

The nature of digital technological tools themselves has significant implications for the educational contexts in which they are employed. As noted by Clark-Wilson, Robutti, and Thomas (2020), a hand tool (e.g., a CAS tool such as *Mathematica* or *Maple*) has its own mathematical system embedded. Its use involves the mastery of commands whose sequencing is different from those utilized in corresponding "by hand" methods when applied to the same mathematics. This has substantial pedagogical implications when, following mathematisation by traditional means, it becomes necessary to invoke technology to advance model development.

In a related vein, classroom implementations of technologies involve incorporating the properties of a tool, as well as taking account of the mathematics that is intended to be learned through its use. In other words, to harness the epistemic value of the tool as an instrument for learning and teaching (Artigue, 2002). The theoretical construct *instrumental genesis*, developed to frame this process (Guin & Trouche, 1999), provides one account of how humans become proficient users of digital tools. No classroom can avoid the consequences of these interactions, and they are very much alive in modelling contexts.

Fast forward two decades to where Jankvist, Misfeldt, and Aguilar (2019) noted (p. 77) that

> …it is not possible for students to describe the origin of unexpected or unstable results experienced in a CAS environment …they are not able to see what part of the instability comes from an unclear formulation of the problem, from a mathematical phenomenon, or from specific methods hard wired into the software.

Implications for modelling are significant, given that it is an activity where formulation of models, and their subsequent solution are complementary overlapping components in any setting.

From the point of view of classroom implementation, the perceived self-efficacy of students adds a further dimension to what they may be able or prepared to do when using technology to engage mathematical content. Characteristically different ways that students operate with technology were identified and captured using a set of metaphors (e.g., Geiger, 2005):

Technology as master: Students (and teachers) are subservient to a technology when their knowledge and usage are controlled and limited by recipe like dependence on its documented properties. If essential mathematical understanding is absent, the user is reduced to blind consumption of whatever output is generated, irrespective of its accuracy or worth.

Technology as servant: Here technology is used as a reliable timesaving replacement for mental or pen and paper computations. Unlike the previous category the user is in control, and 'instructs' the technology as an obedient but 'dumb' assistant.

Technology as partner: At the next level technology acts as a *partner* when shared calculator or computer output promotes peer discussion as students cluster together to compare screens, hold graphical calculators side by side, passing them back and forth to neighbours to compare working, or to resolve mathematical discrepancies.

Technology as extension of self: As the most sophisticated mode of functioning, users incorporate technological expertise as an extended part of their mathematical repertoire. Students may integrate a variety of technological resources into the construction of a mathematical argument so that powerful use of computers and calculators forms an extension of the individual's mathematical prowess.

Such different levels of expertise and confidence have implications for both individual and collaborative modelling success. Evidence from the International Mathematical Modelling Challenge (Garfunkel et. al., 2021) and projects such as described in Galbraith et al. (2018) indicate that some students are able to invoke and employ technology creatively at high levels to enhance their modelling performance. This is by no means universal, and such students have demonstrated abilities consistent with the two upper taxonomic levels of the previously described classification.

Oft cited early research in mathematical modelling (e.g., Treilibs et al., 1980) established that when learning to model, the level of mathematics required needed to be substantially less in demand than that which the same students held in terms of pure mathematical knowledge and skills. In light of later information processing knowledge this could be understood in terms of the need to provide more working memory space for accommodating the demands of learning and applying a new (modelling) process.

Similar awareness is called for when mathematical modelling procedures require technological support for progress. Technology skills needed for addressing modelling problems effectively need to be at the students' fingertips – as an embedded part of a modeller's available repertoire of knowledge and skills. There is a distinction between exploring how students learn mathematical concepts and skills via digital tools (as within mathematics education at large), and the ability to recognize and choose appropriate digital tools and then use them competently in the solution of real-world problems within a total process.

Early uses of technology in modelling focused prominently on the ability to obtain answers within the solution phase of the modelling process. This led to suggested amendments to the fundamental modelling cycle by adding an arrow from box C 'Mathematical Model' in Figure 1 to a new box labelled 'Technology' and thence via another arrow to existing box D – 'Mathematical solution' (Greefrath et al., 2011).

In later work describing modelling research with year 9 students in a Technology Rich Teaching and Learning Environment (Stillman et al., 2007) there was particular interest in how affordances provided by the technology (graphical calculators) were utilised in solving problems in real-world settings. It was confirmed that technology was utilised in some form within each of the stages A to F in Figure 1. Similar outcomes were noted by Geiger et al. (2010) who, following research conducted in a CAS environment, observed that student technology related activity takes place during all stages of the modelling process. In order for these capabilities to be effectively learned and applied in new situations they must first be specifically and effectively taught.

In a comprehensive review, Molina-Toro et al. (2019) identified two different emphases in the use of digital technologies for mathematical modelling within education. The first related to studies which analyze methodological and theoretical aspects of modelling processes; such as where and how students construct mathematical models to solve problems; validate models with the help of software; and construct graphic representations to interpret a situation under study (e.g., Stillman, 2011; Villarreal et al., 2018).

The second theme focused on the interrelation between 'modelling' and technologies, whereby technology is deemed to play a fundamental role in the reorganization of ways of doing 'modelling' and generating knowledge through the process (e.g., Borba & Villarreal, 2005).

The review offered evidence of what new ways of reorganizing certain modelling processes can offer to the broader teaching and learning of mathematics. These are to a large degree, as for Borba and Villarreal (2005), pedagogically determined and most relevant to the vehicle mode, but also contain aspects that are relevant for those whose emphasis lies with modelling treated as content. In asking questions such as: How are modelling processes modified when including technologies? What elements of problems promote the inclusion of technology by students? What do students use technologies for? These authors are more accurately dealing with the impact of technology use on potential modelling routes, rather than underpinning a change to the fundamental problem-solving structure depicted in Figure 1.

Accepting then that in modelling practice technology has a potentially pervasive role, the following illustrative examples have been chosen to indicate ways in which appropriate and inappropriate uses of technology can enhance or impede the productive generation and solution of problems located in real-world settings. Problems A, B, C (Figures 2, 3, and 4, respectively) can be sourced from Galbraith and Holton (2018), D (Figure 5) from Geiger et al. (2021), and E (Figure 7) from Galbraith (2010).

A. Technology use in problem design

> *Waste not want not*
> In February 2016 it was noted that Australians now produce about 50 million tonnes of total waste each year, averaging over 2 tonnes per person. In the period 1996-2015 the population rose by 28% but waste generation increased by 170%. The source indicated that "waste is growing at a compound growth rate of 7.8% /Year".
>
> *Problem*: Estimate the amount of waste that will be generated in Australia over the next 20? 50? 100? years.

Figure 2. Problem *Waste not want not*

A question relevant to implementation is whether information located on a website can be simply trusted for purposes of setting a modelling problem (Figure 2).

A scientific calculator is sufficient to establish that the figures imply average annual growth rates of approximately 1.24% for population, and 5.09% (not 7.8%) for total waste. The website information is identified as suspect with any model building on it resulting in misleading outcomes. Moral – a task designer needs to check background information provided in public sources before using it in problem design, and to convey this insight to students.

B. Technology use to interrogate modelling claims

> *Population projections*
> **Australia's population to hit 23 million Tuesday, 23 April, 2013.**
> Australia's population will reach 23 million people overnight. The Australian Bureau of Statistics estimates that there is a birth every-one minute and 44 seconds, a death every three minutes and 32 seconds, and a new migrant arriving every two minutes and 19 seconds. That means our population increases by one person every minute and 23 seconds – more than 1000 people per day and on track to surpass 40 million within 40 years.

Figure 3. Problem *Population projections*

This statement comes from a press release, which combines an overnight prediction with a forecast that Australia's population will reach 40 million in 40 years (Figure 3).

The (overnight) prediction of a person added every minute and 23 seconds is verifiable from the data, but the juxtaposing of the two forecasts has implications for the 40-year projection. It invites the use of the given births, deaths, and migration data in estimating the 40-year forecast.

The estimate of 40 million in 40 years can be verified (using spreadsheet, or CAS) as a prediction based on the quoted instantaneous parameter values which apply on the given date. But the fractional death rate implied by the figure of one death every 3 minutes and 32 seconds equates to $d = 0.00647\ yr^{-1}$. Its reciprocal as a measure of average lifetime, translates to 154.6 years!

A nice thought, but the figure is clearly misleading if used for long-term projections and needs to be replaced by a more appropriate estimate, such as provided in life-expectancy tables (about 81.5 years).

Likewise, the given immigration data should be replaced by average values which are somewhat higher than those applying on this day. In the absence of other information, it is reasonable to leave the birth data as is. On this basis the forward projections are for a population of around 31 million in 40 years time.

In this instance technology should come into play before model calculations are undertaken, as a means of assessing and amending given parameter values identified as inappropriate for the model's purpose. Evaluation has taken place, not of the model itself, but critically of the data from which it will be parameterised.

C. Technology use to provide strategic options for formulation and solution

Supersize Me
Concern over increasing obesity in developed countries has been growing for years. In 2004 the American Morgan Spurlock created his film-documentary for which on 30 consecutive days he ate three meals daily consisting of nothing but McDonald's food and beverages. Consuming approximately 5000 calories daily he limited his daily exercise to that of the average American office worker. Spurlock, a 32-year-old male who was 188 cm tall and weighed 84.1 kg at the start of the experiment, experienced a weight increase of 11.1 kg over the 30 days.

Problem. Develop and evaluate a mathematical model to describe the weight gain experienced by Morgan Spurlock. (Use the model to explore the respective impacts of calorie in take and exercise on weight gain).

Figure 4. Problem *Supersize Me*

In formulating the model an essential step is to set up the iterative relationship:

Weight today = weight yesterday + [(Intake in calories yesterday − calories used yesterday) (converted to weight)].

Searches will locate data for all quantities on the right-hand side. Choices are now available that impact on the accessibility of the subsequent modelling.

Solutions can be pursued using (a) spreadsheets or (b) geometric series or (c) calculus. Given basic competency with spreadsheets, the approach becomes accessible to year 9 or 10 students. With graphing calculator and/or CAS facility the other methods become efficiently available at a more senior level.

For such examples choice of technology changes the accessibility and mathematical pathways subsequently followed. It is an example of where forethought enables the activation of *implemented anticipation* (Niss, 2010).

D. Technology use to widen options for model evaluation

Is it Worth the Trip?
The rapid change in the price of petrol at the bowser has become common place. Prices can also vary substantially between suburbs towns and states. To minimise costs most effectively is it simply a matter of finding the cheapest price and driving to that location?
Consider the case of Sam. Sam has just finished shopping at the 'The Gap Village Shopping Centre' and realises that s/he's almost out of fuel with only about 4 litres left! Although Sam lives just across the road from the Shopping Centre s/he had promised to return the car with a full tank (or as near to). Sam has a smart phone route finder app as well as a fuel app which shows petrol costs at neighbourhood fuel stations. Sam checks the fuel app and obtains the following price information (in cents/litre) for surrounding petrol stations A to D as follows.

A. 7 Eleven Albany Creek 120.7
B. BP Noonan's Garage 135.7
C. BP Stafford 130.6
D. Puma Everton Park 125.7

Problem: What is the best choice for Sam?

Figure 5. Problem *Is it worth the trip?*

Approaches vary, from those based on specific arithmetical calculations for the given car (junior students), to others that involve algebraic formulations, generalised to apply to any vehicle and for any petrol station location. The spreadsheet below summarises typical output for Sam's car and the respective locations. As can be seen distance travelled, time taken, and fuel remaining on return (F) are all included as decision variables – identified by modellers.

Total cost of fuel alone (T) points to A as the preferred choice. But when additional time and distance are factored in, the tiny total saving for A over D of 64 cents is swamped by real-world practicality. Puma (D) emerges as the better real-world option. The *mathematical* solution that minimises fuel cost is not the best solution to the problem in *real-world* terms.

A spreadsheet (like the one in Figure 6) not only provides for multiple mathematical outcomes to be calculated instantaneously. It also displays comparative outcomes so that real-world criteria are facilitated in evaluating the worth of strictly mathematical calculations. Again, forward thinking at the outset of (and during) the problem solution will suggest how technology may both facilitate mathematical calculations and put them into perspective in terms of the total problem.

	A	B	C	D	E	F	G	H
1	Petrol Station Locations	(D) distance (km)	(P) price (cents/l)	Parameters	V (litres)	T ($)	F (litres)	M (min)
2	A	16.7	120.7	(L) fuel in tank at start (litres)	39.19	47.30	40.81	33
3	B	5	135.7	4	38.36	52.05	41.64	10
4	C	9.8	130.6	(c) fuel economy (km/litre)	38.70	50.54	41.30	20
5	D	10	125.7	14.08	38.71	48.66	41.29	20
6	E			(C) tank capacity (litres)				
7	F			42				
8				(S) speed limit (km/hr)				
9				60				

Figure 6. Example of spreadsheet concerning the problem *Is it worth the trip?*

E. Technology or data sources – where does authority lie?

The focus here is on handling tensions between competing authorities, for example machine output, and modelling integrity. Doerr and Zangor (2000) referred to student preference for calculator output over contextual reality, whereby students insisted on working with multiple decimal places on problems whose data involved using crude measuring devices. This issue emerges whenever the two *authorities* compete: output from a technological device, and real-world data.

Referring to the problem in Figure 7, using the data and real-world knowledge that a year consists of 365 days (approximately), with summer and winter solstices on Dec 21, and June 21 leads to model equations such as:

$$\text{Melbourne: } y_M = 730 + 158\cos\left(\frac{2\pi}{365}(x+11)\right)$$

$$\text{Brisbane: } y_B = 730 + 106\cos\left(\frac{2\pi}{365}(x+11)\right),$$

where x is measured in days, and y in minutes. These suffice to address the modelling problem, and to suggest refinements.

Of interest is the potential impact of technology choice and use. Given a request to fit a model to data, it has been observed that some students proceed to use every function available on their graphical calculator. The criterion is to find whatever formula minimises R^2, no matter how irrelevant the function might be in terms of the data source.

Living Daylights

The amount of daylight affects many aspects of life. It helps determine growing seasons, impacts strongly on the tourist industry, and is a factor influencing personal lifestyles and choices. (For present purposes, daylight means the time between sunrise and sunset – aspects such as twilight are not included.) Data for a year, at 4-weekly intervals, sourced from a web page are entered in the table below for Melbourne and Brisbane.

Dates	Day no	Melb Mins	Bne Mins	Dates	Day no	Melb Mins	Bne Mins
01-Jan	0	884	831	15-Jul	196	588	634
29-Jan	28	846	806	12-Aug	224	634	665
26-Feb	56	784	765	09-Sep	252	697	706
25-Mar	84	716	719	07-Oct	280	764	751
22-Apr	112	650	675	04-Nov	308	828	794
20-May	140	598	640	02-Dec	336	876	826
17-Jun	168	573	624	30-Dec	364	884	831

Problem: Plot the given data on a spreadsheet and develop periodic functions to model the data for Melbourne and Brisbane. Draw graphs showing both the original data and model values on the same page. How good is the fit?

Using the model equations find an estimate of the total number of minutes of daylight in Melbourne and Brisbane over the year and the average minutes per day.

Queensland markets itself as the Sunshine State and aims to attract southern visitors, particularly in winter. How much extra daylight is there in Brisbane during that period of the year when Brisbane has more daylight than Melbourne?

Figure 7. Problem *Living daylights*

In this problem it is possible to use the sinusoidal regression facility of a graphical calculator to fit a function to the data without any regard to the seasonal variations that underpin them. If this is done, the function with minimum R^2 corresponds to a year length of 377 days, with the summer solstice on December 15. Further, if the quartic polynomial function option is chosen for regression on the TI-83 (or later) graphical calculator R^2 comes out as 0.9987 which is the same value as for the cosine function noted above. This latter outcome occurs of course because a single period (year) of the daylight variation has been considered. Are they therefore equally valid? Of course not, but the point is underlined that in terms of choosing processing options made available by technology, choices must be made according to known properties of the real-world data involved, and not on the basis of menu choices available on electronic devices.

In summary, we have reviewed ways in which technology can be called upon to facilitate mathematical modelling that is prioritised as real-world problem solving. In this endeavour authority for action resides first in maintaining the authenticities of problem design and the problem-solving process. Finding uses for favoured technology is not a priority, except in so far as familiarity with specific devices, programs, and modes of working in classrooms, can be invoked to achieve outcomes more effectively. Opportunities for misuse or oversight abound when motivation is driven primarily by interests in particular technologies, rather than in the way that they should be used critically to enhance the modelling process and its outcomes.

Simulation: when technology is indispensable to modelling

In this final section we consider the situation in which effective modelling is inaccessible without technology. Such examples involve non-linearity and simultaneity among model relationships meaning that simulation is required for solution purposes.

Two families of models are identifiable here. Firstly, where equations exist and are relevant to a problem, but assumptions and parameter values need to be articulated and assigned. Simulation using CAS software or spreadsheets is often a suitable approach to take to the solution of problems of this type. An example of this type is provided through the modelling of aspects of the spread of pandemics such as COVID-19 (e.g., Galbraith, 2020).

Secondly are problems where variables must first be created, and the model equations themselves constructed, parameterised, and solved by simulation. Such problems are not accessible by methods supported by standard CAS packages. That problems of this type lie beyond the capabilities of modellers equipped with conventional tools has been illustrated in a paper written around the International Mathematical Modelling Challenge (Garfunkel et al., 2021). The problem set in the 2019 Challenge was the following:

> What is the Earth's Carrying Capacity for Human life? 1. Identify and analyze the major factors that you consider crucial to limiting the Earth's carrying capacity for human life under current conditions. 2. Use mathematical modelling to determine the current carrying capacity of the Earth for human life under today's conditions and technology. 3. What can mankind realistically do to raise the carrying capacity of the Earth for human life in perceived or anticipated future conditions? What would those conditions be?

The authors noted that a common approach was to initially consider each chosen factor individually, determine the carrying capacity based on this factor, and then identify the actual carrying capacity as the smallest of these - the limiting factor being when all that resource was used up. They noted that many teams limited their model of carrying capacity to very simple ones that did not take into account that different critical resources are not independent. The recognition of some interdependence of resources was identified as a discriminator of better attempts, but no submitted solution was adequate in terms of the Challenge question.

This outcome is not surprising. The challenging nature of the problem originally emerged from the widely misunderstood and/or misrepresented study published as *Limits to Growth* (Meadows et al., 1972). Subsequently further modelling has been used in producing updated accounts such as *Beyond the limits* (Meadows et al., 1992) and later studies. The modelling methodology System Dynamics was invented specifically to enable problems of this nature to be addressed. We now describe the methodology and illustrate its modelling characteristics by considering the problem of providing potable water for a population that is expanding into a warmer global environment, with limited available water reserves. This features the treatment of the interdependence of variables in a simpler modelling context than the carrying capacity problem, for which the same approach needs to be taken.

Modelling problems in complex systems through system dynamics

Complex systems are playing a more prominent role in the *wicked problems* that are confronting scientific, economic, and political decision makers. Problems like spreading pandemics, economic recession, environmental degradation, re-settling millions of homeless people, and increasing political polarization are some of the problems that require analysis defined by attention to interconnections, feedback, non-linearity, and delays. For these types of modelling problems, the variables are not obvious up front, and their choice, definitions and the relationships driving their growth and decline are challenges a modeller must engage with.

System dynamics (SD) was invented by Jay Forrester at MIT in the middle of last century, as a means to address problems that feature non-linearity, and simultaneity, characteristics that can only be addressed through simulation (Forrester, 1969). It addresses the need to identify variables of relevance, and build equations relating them, before an approach to their solution through simulation can be contemplated. The development of the Stella software in 1985, followed later by parallel software (Powersim and Vensim), provides a visual, icon-based approach which has brought the fundamentals of system dynamics modelling within the reach of secondary students (Fisher, 2017, 2018); in addition to its fundamental role as a powerful problem-solving agent for addressing real-world problems as illustrated here. For examples that build explicit links between SD approaches and traditional differential equation methods see Galbraith and Fisher (2021).

The tools

There are four icons that form the basis of the modelling toolkit. Stocks (rectangles) are accumulators, similar to integrals in calculus, that are used to represent the main functions we want to track over time. Their levels rise and fall in response to rates of change (inflows and outflows). The net flow for a stock represents its overall rate of change or first derivative in calculus terms.

Connectors and converters (icon-based descriptors) are names for components that are readily related to familiar mathematical structures (Figure 8). In linking variables and parameters that are related, connectors activate dependencies of one component on another in the same way that independent variables feed into the expression of their dependent counterparts. During simulation connectors transmit the updated values calculated at each timestep. Converters contain the mathematical formulae and parameters which create numerical output at each timestep, and which collectively combine to define the flows at that point in time.

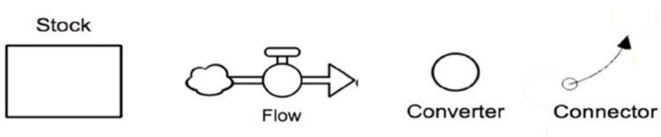

Figure 8. The four icons used to create system dynamics models

Example: The potential impact on population of reductions in the amount of available potable water as a result of increasing global temperatures

First stage: global population growth since 1950. Since 1950 the global population has maintained a net growth rate of about 1.6% per year in spite of the reduction in overall birth fraction and overall death fraction over those years. The SD software calculates stock values recursively.

For global population. In differential equation format we have

$$dP/dt = bP - dP = (b - d)P$$

Approximating with $\delta t = 1$, the recursive formula used by the software is:

$$P(t) = P(t\text{-}1) + (b - d)P(t\text{-}1)$$

Assuming a birth fraction of 0.038 and a death fraction of 0.022 and a beginning global population of 2.5 billion in 1950 we have:

$$P(1) = 2.5*109 + (0.038)(2.5*109) - (0.022)(2.5*109) =$$
$$= 2.5*109 + (0.016)(2.5*109) = 2.54*109$$

$$P(2) = 2.54*109 + (0.016)(2.54*109) = 2.581*109, \text{etc.}$$

The SD population model and its graphical output are shown in Figure 9.

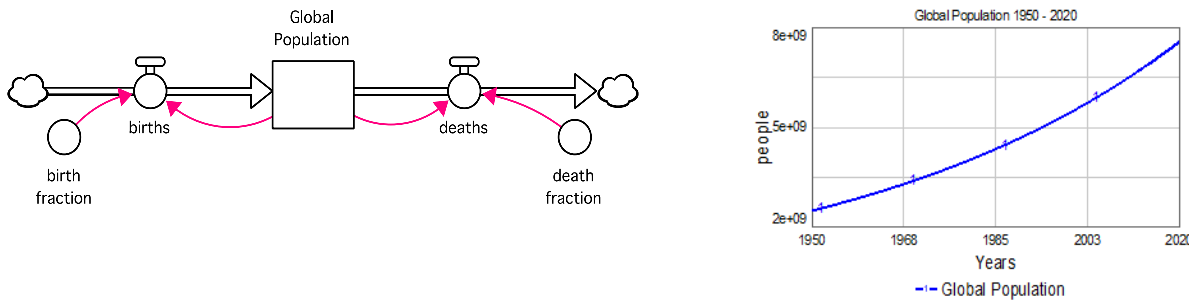

Figure 9. System dynamics model for global population growth (left) and model output (right). (Numerical definitions of each icon have been superimposed on the model diagram for convenience)

Birth fraction is given by the average value of (births/year)/person estimated across the globe. Similarly, death fraction equals the corresponding average value of (deaths/year)/person. The unit for both is 1/year. As noted above, while both birth and death fractions have changed, their net difference has remained essentially the same since 1950.

Looking at the equations above Figure 9 for global population:

(a) The calculation for the first year uses estimates of the actual birth and death fractions at that time (1950).

(b) For the remaining years, calculations are based on the knowledge that the *difference* between these values stays constant (at 0.016) rather than the individual values.

(c) The power of the method can be seen in that calculations are set up to work if the individual values change independently, by using the method of the first year. The flexibility of the SD approach is underlined as stock levels continuously accommodate to individual changes to inflows and outflows that occur over time.

(d) Values of population are calculated recursively at integral values of time (years). When fractional timesteps (e.g., 0.2) are used to approximate more closely to continuous variations the method is the same (timestep by timestep) with the mathematical approach aligned to numerical integration.

Second stage: expanding the model. To model the impact of changes in the availability of potable water we introduce accessible potable water as a stock, together with another stock global temperature which together with population represent major factors in changing levels of available water. Bearing in mind contemporary arguments around climate change let us consider how global CO_2 emissions could influence global temperature. In Figure 10 we are assuming the global temperature was already increasing, due to CO_2 emissions, in 1950, by twice the ability of the planet's atmosphere to neutralize the effect of CO_2 emissions. The values used in this model are readily available on the web (Macrotrends and NASA). In the descriptions that follow, emphasis is on the reasoning behind the development of model structure.

On the left is the SD model structure showing the influence of CO_2 emissions on global temperature change and the influence of the amount of potable water on population death fraction. Note that the model definitions for each icon are superimposed on the model diagram.

a = 3.253+ramp (0.0276),
b = 0.018-ramp (0.0005, 2005, 2025),
g = (492*(1+percent increase in water consumption each year) ^ (TIME-1950)) * effect of water available on consumption per person.

Ramps represent a constant rate of increase or decrease in the value to which it is being applied.

On the right (top) of Figure 10 is shown the graphical definition of the effect of available potable water on the death fraction. On the right (bottom) see the model output for Business as Usual.

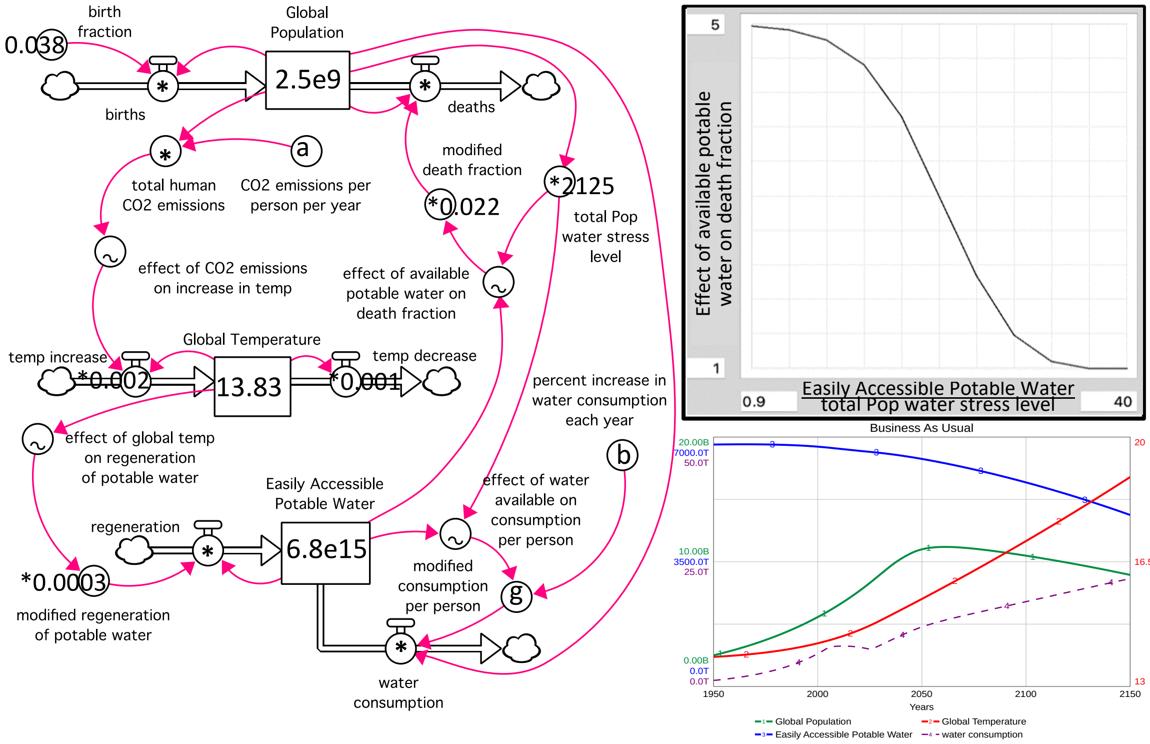

Figure 10. Expanded diagram for stage 2 of the model

The most interesting parts of the model are the four components that can be defined graphically. They are identified by tilde symbols in the converters and define respectively:

(i) influence of CO_2 emissions on global temperature,

(ii) influence of available potable water on death fraction,

(iii) influence of global temperature on regeneration of potable water supply,

(iv) influence of the amount of available potable water on consumption per person.

The representation and justification of such relationships is central to a modeller's activity in the formulation phase of an SD model. For (i) a linear relationship is used to approximate the effect of CO_2 emissions in terms of increasing temperature (Leduc et al., 2015).

For (ii) an S-shaped functionality was chosen (see Figure 10). We assume there has been no impact of available potable water (shortage) on death fraction up until 2020, and such change in the near future is estimated as negligible. The corresponding output of the graphical converter is (the multiplier) 1. As potable water begins to decrease below the 2020 level, its influence on the death fraction of population increases, slowly at first, then rising sharply as the surplus depletes, until the modified death fraction is 5 times the original death fraction when the potable water available per person falls to 90% of the global population water stress level. Note that water stress level is defined as the minimum amount of water needed per person per year, for normal consumption and per capita agricultural and industrial needs. Values below stress level indicate water is becoming increasingly scarce (Damkjaer & Taylor, 2017). In this model, in 2020 the per capita available potable water is 40 times the water stress level. As the future global population grows the amount of potable water needed per person increases exponentially, for Business as Usual scenario.

For (iii) (not shown diagrammatically), an S-shaped relationship is again deemed appropriate. The 1950 temperature is normalized to 1. The multiplier representing the proportion of potable water that can be regenerated at this temperature has the value 1 and retains a value close to this number until the temperature rise beyond this figure begins to reduce the capacity to fully replenish stocks. As the temperature increases further the multiplier continues on a faster (non-linear) numerical descent reflecting increasing impairment of the ability to replenish depleted stocks. The domain here is from 1 to 1.3 (for temperature) representing an increase of average global temperature from 13.83^{o} C (in 1950) to approximately 18^{o} C, at which point the ability to regenerate is reduced to 70% of its original capacity.

For (iv), the influence of the amount of water available on consumption per person has domain 0.5 to 40 and range 0.5 to 1. 40 represents the 2020 value (potable water available per person is 40 times the water stress level). When the ratio of potable water available per person/water stress level is 20 or higher there is no reduction in yearly consumption of water. But as the ratio falls below 20 the yearly consumption of water per person is impaired and starts to fall quickly, decreasing at an increasing rate, to half of the normal consumption per person for a given year.

The design of this type of graphical function definition follows specific rules that are part of the system dynamics modelling method. They are guided by a mix of data trends, and advice of experts when available, with the shape of the graph an educated *guess* in terms of its mathematical features as illustrated in the foregoing discussion. The shape of the graph must be supportable as a reasonable approximation to what would be expected in terms of real-world movements as the independent variable and dependent *multiplier* value fluctuate away from their initial equilibrium (1,1) position.

The model is tested for sensitivity, by examining the impact that small changes in functional shape and parameter values have on output curves such as those displayed in Figure 11. The variables of a robust model will feature similar behaviour modes when inputs are varied within ranges legitimated by assumptions, although numerical values will naturally alter.

The choices described above demonstrate another fundamental principle that underpins SD modelling. It is not acceptable to omit a process of known significance on the grounds that *hard data* are not available. Processes are included because they are recognized as important real-world factors, not because they are easy to define in terms of quantitative data. To *omit* such a process on the grounds of insufficient data is not to omit it at all – but to include it with an assigned weight of zero. This is a far more serious structural error than getting the shape of an effect correct, but its detail approximate.

Testing policies to address future problems emerging on account of diminishing accessible freshwater supply. We tested policy effects on reducing anthropomorphic CO_2 emissions to zero over time and reducing the amount of water consumed per person over time, assuming they might add to the time we have to seek additional solutions for global environmental crises. Figure 11 (output from the SD model shown in Figure 10 with policies implemented) shows model output respectively for an emission reduction policy only; a reduction in the growth of water consumption per person per year policy only; and the implementation of both policies together It was assumed that the effect of a rise in global temperature would impact the death fraction most predictably through water availability and water consumption crises that are generated in the context of an increasing population. If rising temperature alone is deemed to exert a direct impact on death fraction, its influence can be added by means of a direct link to the modified death fraction. Note that the potential impact of global temperature on death fraction has been predicted to occur after a global temperature of 16.5^{o} C is reached, which is marked by the centre horizontal gridline on each grid (Figure 11).

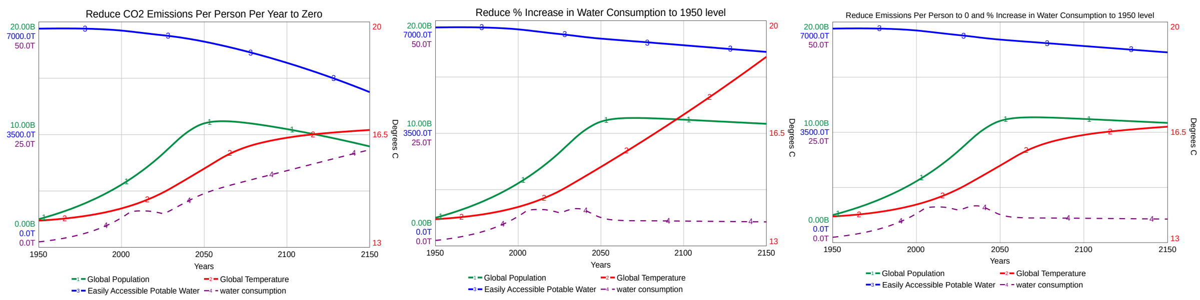

Figure 11. Policies starting in 2020: (left) CO_2 emissions reduced, (center) growth in consumption of water per person reduced, (right) both of the previous policies applied together

Desalination plants, critical to water supply in countries like Saudi Arabia, have been growing at a linear rate of almost 500 per year since 1985. While providing a hedge against the threat of diminishing availability of potable water for those countries, they do not represent a long- term global solution.

> As we move toward models that represent people, their decisions, and their reactions to the pressures of their environment, it is well to keep in mind these relative rather than absolute measures of model utility. The representation need not be defended as perfect, but only that it clarifies thought, captures and records what we do know, and allows us to see the consequences of our assumptions, whether those assumptions be perceived ultimately as right or wrong. A model is successful if it opens the road to improving the accuracy with which we can represent reality. (Forrester, 1969, pp. 4-5)

Reflection. The example chosen illustrates the main features of SD modelling as a real-world problem-solving process. Although it is not the purpose here, we note that secondary school students have built and explained SD models similar to this one (e.g., Fisher, 2018).

While the modelling here (involving three stocks) has used a commercial version of Stella there is a free, web-based *lite* version of the software (available at exchange.iseesystems.com). The software, Stella Online, only allows a maximum of two stocks to be used in a model, and it allows a total of 12 icons (including the two stocks, excluding connectors in the count) and up to three graphs. It is possible to create the model shown in Figure 10 (not including potable water, but including a link from global temperature to modified death fraction) in that free software. It can be noted that the documentation, including examples, provided with the free software is excellent, and provides elaborations of modelling specifics that space has precluded here.

Afterword

This paper has had two main purposes. Firstly, to selectively overview the role of technology as an agent within mathematics education, with specific reference to its contribution, actual and potential, to the development of mathematical modelling as real-world problem-solving. Research has demonstrated that technology, when used effectively in modelling, is invoked at every stage of the modelling process.

We have drawn attention to some of the issues involved in using technology effectively for modelling purposes, and illustrated, through examples, how it can both facilitate and mislead when employed in modelling contexts.

Secondly, we have illustrated how technology can enable the development of models that are inaccessible when only hand methods of solution are available. Such problems involve non-linearity and simultaneity among model relationships meaning that simulation is required for solution purposes. A problem of contemporary interest – the projected impact of continued global warming on a global population making increasing demands on the supply of potable water is developed as an example using the increasingly accessible methodology of System Dynamics modelling. It is noted that System Dynamics would enable the sustainability problem set for the 2019 International Mathematical Modeling Challenge to be treated more decisively.

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