Mathematical modelling routes supported by technology in the learning of linear algebra: a study with Costa Rican undergraduate students

Rotas de modelação matemática apoiadas pela tecnologia na aprendizagem da álgebra linear: um estudo com estudantes universitários da Costa Rica

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Resumo. A modelação matemática é um contexto relevante e produtivo para o ensino da álgebra linear, favorecendo a aprendizagem de conceitos e competências de modelação dos estudantes, comumente negligenciadas na matemática universitária. Este estudo visa caracterizar rotas de modelação realizadas e competências postas em prática por estudantes universitários da Costa Rica, numa disciplina de álgebra linear, ao resolverem uma tarefa de modelação matemática envolvendo conceitos de conjunto gerador e base de um subespaço vetorial, usando uma folha de cálculo. A recolha de dados incluiu observação participante das discussões dos estudantes e suas resoluções escritas, bem como os ficheiros Excel produzidos. A análise revelou rotas de modelação dos estudantes incompletas e completas, nas quais utilizaram conceitos de combinação linear, subespaço gerado, base canónica e não canónica para construir modelos. O uso da tecnologia influenciou o modo como os alunos obtiveram resultados matemáticos e reais extraídos do seu modelo matemático. Nas rotas completas, o modelo computacional impulsionou transições no ciclo de modelação entre

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resultados computacionais, matemáticos e reais, melhorando os processos de modelação. Complementarmente, o Excel desempenhou duas funções principais, calcular e simular, permitindo aos grupos que o utilizaram avaliarem a eficácia dos seus modelos matemáticos e computacionais e validá-los no confronto com a situação real.
Palavras-chave: álgebra linear; estudantes universitários; rotas de modelação matemática; modelo computacional; folha de cálculo.

Abstract. Mathematical modelling is a significant and productive context for teaching linear algebra, favoring students' learning of concepts and modelling competencies, often neglected in undergraduate mathematics. This study aims to characterize the modelling routes performed and the competencies put into practice by Costa Rican undergraduate students in a linear algebra course, when solving a mathematical modelling task involving the concepts of spanning set and basis of a vector subspace, using the Excel spreadsheet. Data collection included participant observation of the students' discussions, their written work on the task, and the Excel files produced. The analysis reveals incomplete and complete modelling routes performed by the students, in which they used the concepts of linear combination, span, standard and non-standard basis to build models. The use of technology influenced the way students obtained mathematical and/or real results from their mathematical model. In complete modelling routes, the computer model has driven several transitions within the modelling cycle, namely between computational, mathematical, and real results, enhancing the modelling processes. In addition, the spreadsheet performed two main functions, calculating and simulating, allowing the groups who used it to assess the effectiveness of their mathematical and computer models and validate them against the real situation.
Keywords: linear algebra; undergraduate students; mathematical modelling routes; computer model; spreadsheet.

Introduction

The study of linear algebra is an integral part of several undergraduate courses, in areas ranging from Science and Engineering to Economics and Management. It is a fundamental area of mathematics, not only because it involves concepts that are used in subsequent mathematical topics, but also because many of such concepts and methods are suitable for modelling real-world phenomena and for solving real problems (Costa & Rossignoli, 2017). The central role that linear algebra undertakes, especially in higher education, along with the difficulties that many students reveal with this subject, has led educators and researchers to emphasize the need to promote learning with understanding, seeking to give meaning to linear algebra concepts that are often seen as quite abstract. Teaching proposals that embrace the meaningful learning of linear algebra concepts accentuate the role of contextualized situations, namely by including mathematical modelling and applications in the study of linear algebra (Cárcamo, Gómez, & Fortuny, 2016; Wawro et al., 2012). As it already happens in other countries, the linear algebra course for undergraduate students of

Engineering and Sciences offered by a Costa Rican University considers the idea of addressing linear algebra contents as mathematical models. Moreover, the use of technological tools (including CAS) is similarly suggested as a learning enrichment strategy.

Mathematical modelling has a long tradition and is deeply rooted in higher education, where studies concerning its implementation in mathematics courses have revealed learning improvements (Alsina, 2007). From a cognitive perspective, mathematical modelling is conceived as a sequence of actions and processes taking place as the modeller creates and uses a mathematical model to achieve a solution to a problem in the real world. The modelling cycle provides a way of representing the trajectories followed by the students in their modelling activity, known as the students' modelling routes (Borromeo Ferri, 2018), and offers a view on the students' cognitive modelling processes and their difficulties in constructing models. However, the research on the student's modelling under a cognitive perspective is still scarce (Borromeo Ferri, 2007); furthermore, the effects of technology use in mathematical modelling are not fully investigated and they have not been a focus of attention in studies directed at the student's modelling routes (Borromeo Ferri, 2018; Greefrath, Hertleif, & Siller, 2018).

In view of the above, we consider the pertinence of our study, which is based on a teaching proposal aiming at promoting Costa Rican undergraduate students' modelling activity with linear algebra, in which they can use technological tools. The central aim of the study is to characterize the modelling routes performed and the competencies put into practice by students in a linear algebra course, when solving a mathematical modelling task with the use of the Excel spreadsheet.

The research endorses a cognitive perspective consistent with a detailed analysis of the cognitive processes carried out by the modellers in their distinct modelling routes. This is relevant in the Costa Rican context where the cognitive perspective on modelling, namely the study of students' modelling routes, is absent. As the university course program (named MA1004) encourages the use of technological tools in linear algebra problems (Sánchez, 2019), the current study also investigates the consequences of technology as part of modelling a real-world situation, where the concepts of spanning set and vector subspace are involved.

Theoretical framework

The learning of vector space concepts considering mathematical modelling and considering the use of technology

According to Rensaa (2017), the more common difficulties that students reveal in learning linear algebra are linked to the proof-centred pedagogical approach, the formalism involved in theoretical concepts, and the need to coordinate the practical and the theoretical character of ideas and concepts of linear algebra. Additionally, students seem to be more open to using mathematics in a practical way and are mostly interested in seeing the application of mathematics in real problems (Rensaa, Hogstad, & Monaghan, 2020).

Researchers and teachers have become increasingly aware of those difficulties and have proposed pedagogical approaches that move away from the formal teaching of linear algebra, namely by integrating mathematical modelling or using technology in their lessons, as highlighted by Stewart, Andrews-Larson, and Zandieh (2019). In the study by Rensaa, et al. (2020), a questionnaire was applied to experienced university teachers, aiming to identify their perspectives on the teaching and learning of linear algebra concepts. The results showed a consensus on the importance of involving students in mathematical modelling and on the use of relevant technology to solve modelling problems in linear algebra. However, few teaching proposals include modelling tasks and technology in an articulate way and the role of that articulation for the learning of algebraic concepts remains unclear.

Regarding specific studies about the learning of vector spaces concepts, Cárcamo et al. (2016) report on the use of technology (Excel spreadsheet) in mathematical modelling environments to introduce the concepts of spanning set and span. They concluded that the pedagogical proposal contributed to the construction of the concepts of spanning set and span and helped the students to develop higher-order cognitive skills related to the modelling situation, mainly during the construction of models and the interpretation of mathematical results. However, the emphasis of the study on hypothetical learning trajectories and emergent models made the analysis of the role of technology less visible, which suggests the need for a more distinct focus on the role of the spreadsheet for the learning of concepts and for the development of modelling competencies.

Other studies have focused on integrating modelling and technology for the teaching of specific concepts on vector spaces. For example, Trigueros and Possani (2013) present a modelling task related to economics for exploring concepts such as linear combination and linear dependence and independence of vectors. The results showed the students' initial difficulties in working on the problem but also revealed that through exploring graphical and algebraic representations of linear algebra concepts they were able to describe the real situation. Dogan-Dunlap (2010) also conducted a study using an online graphic calculator and reported that the visual representations provided by this tool promoted a richer understanding of linear independence, namely when students connected geometric representations with arithmetic and algebraic.

The above studies emphasise the importance of the use of multiple representations when students work in modelling and technology combined environments. Despite such evidence, they do not discuss the possible routes that students can follow when solving the tasks by using technological tools.

Mathematical modelling competencies

A prominent concept related to the abilities that students must develop to move through the different phases of the modelling cycle is that of mathematical modelling competency (Maaß, 2006). Niss, Blum, and Galbraith (2007) state that mathematical modelling competency is:

> the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyse or compare given models by investigating the assumptions being made, checking properties and scope of a given model. (p. 12)

Mathematical modelling competency refers to the abilities that are required to carry out the different sub-processes involved in the modelling process, associated to the different phases of the ideal modelling cycle. It should be noted that such sub-processes are variously highlighted and prioritized, depending on the aims of modelling in the educational context and according to a possible perspective on mathematical modelling, such as the pragmatic, the pedagogical, or the psychological, for example (Kaiser & Sriraman, 2006). Our study takes a pedagogical and a psychological stance in the analysis of the student's cognitive processes since both perspectives consider the development of modelling competencies and the learning of mathematical concepts and procedures as fundamental aims of mathematical modelling in education. In focusing on students' modelling routes and their modelling competencies when they use technology, we pay special attention to the role of technology in the modelling cycle.

As suggested by Greefrath (2011), the integration of technology in the modelling process allows taking advantage of its multiple affordances in different phases of the process, from the real situation to the mathematical model, meaning that more opportunities are available to: investigate, experiment, visualize, calculate, simulate, control, evaluate, etc. Therefore, for each sub-process taking place in the transition between phases of the modelling cycle, the use of technology may bring new possibilities to the modeller, as depicted in Figure 1, and positive implications in the development of modelling competencies are expected (Siller & Greefrath, 2010). Whether a student uses a technological tool in his/her modelling process depends on how he/she identifies the advantages or the need to use it. In addition, the necessary skills to use technology to perform a specific activity also affect its use or suppression. This has implications on the student's modelling route, as it can extend or reduce the ability to obtain a solution to a real-world problem.

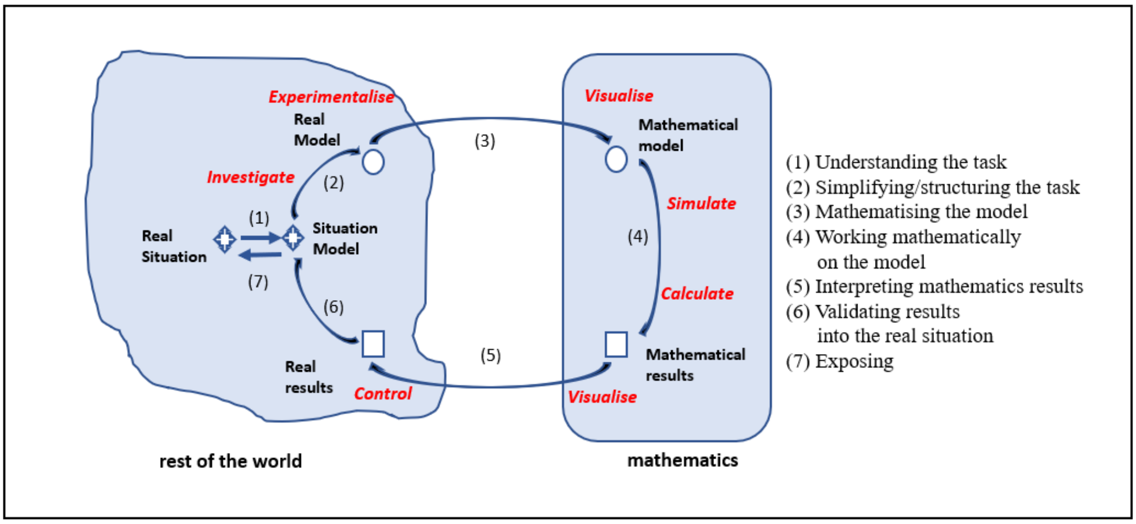


Figure 1. Possible uses of technology in the modelling cycle (based on Blum and Leiß, 2007, and Greefrath, 2011)

It should be acknowledged, however, that for the student's successful work on real-world problems, other mathematical competencies are required, such as, representing mathematical objects, justifying simplifications and assumptions made, in combination with social skills, including those linked to cooperative work (Niss et al., 2007). According to Maaß (2006), there are several mathematical modelling sub-competencies associated with the main modelling sub-processes, as shown in Table 1.

Table 1. Mathematical modelling sub-competencies (adapted from Maaß, 2006)

Modelling sub-competencies	Competency to...	Sub-processes
To understand the real problem	understand the task statement, to conceive a real world situation enclosed in the task and to develop a mental situation model.	(1) Understanding the task
To set up a model based on reality	make assumptions about the problem and simplify the situation; recognize quantities that influence the situation; identify key variables; construct relations between the variables; look for available information and differentiate between relevant and irrelevant information.	(2) Simplifying/ Structuring the task
To set up a mathematical model from the real model	mathematize relevant quantities and relations; simplify relevant quantities and relations to reduce their number and complexity; choose appropriate mathematical notations and represent situations graphically.	(3) Mathematising
To solve mathematical questions within this mathematical model	use heuristic strategies such as division of the problem into smaller problems, establishing relations to similar or analogue problems, rephrase the problem, consider the problem in a different form, vary the quantities or the available data, etc.; use mathematical knowledge to solve the problem.	(4) Working mathematically on the model

To interpret mathematical results in a real situation	interpret mathematical results in extra-mathematical contexts; generalize solutions that were developed for a special situation; use appropriate mathematical language to communicate about the solutions.	(5) Interpreting mathematical results
To validate the solution	critically check and reflect on solutions; review some parts of the model or go through the modelling process again if solutions do not fit the real situation; reflect on other ways of solving the problem or check if solutions can be developed differently; generally, question the model.	(6) Validating results in the real situation

The several sub-competencies described are essential in the student's modelling activity, which indicates a highly cognitive demand in modelling tasks but also their strong potential in contributing to the learning of concepts and procedures (Blum, 2015).

Modelling routes: transitions between the mathematical world, the rest of the world, and the world of technology

Depending on the goals of mathematical modelling in the classroom, the focus may be on developing all or some of the previous described sub-competencies. From a cognitive perspective and for the study of students' learning, each of the presented modelling sub-competencies is interesting, but the same is true from a didactical point of view when the teacher intends to support students' understanding of linear algebra concepts. This supports the relevance of identifying different modelling routes developed by the students (Borromeo Ferri, 2018). As pointed out by Borromeo Ferri (2018), "the individual starts this process during a certain phase, according to his or her preferences, and then goes through different phases several times or only once, focusing on certain phases and/or ignoring others" (p. 30).

When the transitions between the sequential phases occur only once and following the direction of the ideal modelling cycle, we have a so-called linear modelling route. On the other hand, when the transitions involving one or more phases take place several times, we consider it as a non-linear modelling route. It is also known that the modelling routes are influenced by at least three factors (Borromeo Ferri, 2018): (i) mathematical thinking styles (visual, analytical, integrated); (ii) mathematical competencies; and (iii) extra-mathematical experiences and knowledge. Thus, depending on his/her modelling competency, the student's route may show several or few transitions between phases. This means that the modelling process is mostly not linear (in the sense of sequentially pre-defined), but an entangled process influenced by a set of metacognitive activities that take place during the student's work on the task (Galbraith & Stillman, 2006), including the possibility of using technology.

Siller and Greefrath (2010) suggest that working on modelling tasks supported by technology leads to the simplification of mathematical procedures in the modelling process. Their proposal undertakes a modification of the modelling cycle, where the presence of the technological world adds two new phases to the modelling cycle (Figure 2).

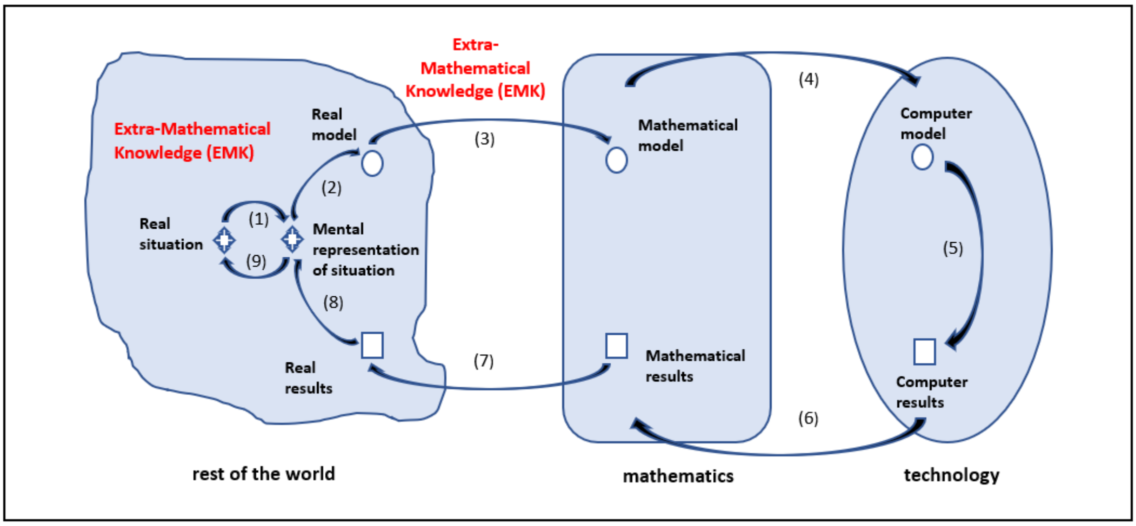


Figure 2. Modelling cycle extension incorporating technology (adapted from Siller and Greefrath, 2010)

In this way, the ideal modelling cycle incorporates a technological world and considers a possible step after the formulation of a *mathematical model* (3). So, the individual can choose to use some technological tool to work on the mathematical model (4) to obtain a *computer model*, and later use the technological tool (5) to obtain *computer results*, which must be decoded (6) to become *mathematical results*. However, if the individual chooses to obtain mathematical results without using technology, there will be a path that takes place only between the real world and the mathematical world. Therefore, the modelling cycle proposed in Figure 2 allows to infer that there are at least two possible ideal modelling routes that the student could follow when building a mathematical model: ignore the technology or use it to obtain mathematical results.

Greefrath et al. (2018) emphasize the need of further research on mathematical modelling scenarios addressing complex problems using technology, especially if the technology has a key role in the development of the modelling process and thus the modeller is compelled to use it. As such, a meaningful description of the role of technology in modelling must consider several uses of digital tools in different phases of the modelling cycle, and therefore must not represent a pure addition as Figure 2 could imply.

Borromeo Ferri (2018) suggests that students should get an idea of how digitals tools, such as Excel, MATLAB, or CAS, may be used in mathematical modelling, for instance, by learning that a spreadsheet can be used to work on real problems to model a specific situation through discovering and representing relations between its variables. In the same vein, a spreadsheet offers the possibility of exploring concepts, such as the basis of a vector subspace and the span of a subspace in $\mathbb{R}^n$ from a numerical viewpoint. In general, a spreadsheet allows students working with different representations in solving a contextual problem, which suggests that mathematical modelling can benefit from its use, similarly to what happens, for example, with algebraic word problems, in which the tool helps to create bridges between numbers and algebra (Haspekian, 2005). Also, authors such as Chaamwe and Shumba (2016) refer the advantages of the Excel spreadsheet in manipulating formulas and perform different mathematical operations, in addition to providing alternative solution processes that allow a greater understanding of the concepts involved. So, there are several ways to work on linear algebra concepts with Excel, possibly exerting an influence on the modelling routes performed, depending on the student's competencies to create meaningful representations, algorithms, and computations, with a modelling purpose.

Methodology

Study context and research questions

This study is based on a teaching experiment, carried out in a linear algebra course at the University of Costa Rica, aiming at improving the students' learning of linear algebra concepts and promoting modelling skills. This intervention involved several lessons where the students solved five modelling tasks and five purely mathematical tasks, in alternate weeks. In the purely mathematical tasks, the students had to use some of the technological tools available in the course but in the modelling tasks its use was mostly optional.

The research presented focuses on the students' work in solving one of the modelling tasks proposed during the teaching experiment. The following research questions were set, according to the aim of the study:

(1) What are the types of the modelling routes performed by the students while solving the task?

(2) Which modelling sub-competencies do the students reveal in their modelling routes while solving the task?

(3) In which steps of the students' modelling routes is the technology used and what are the advantages they take from it?

The modelling task

The proposed task was adapted from Cárcamo et al. (2016) and was expected to engage the students with the concepts of spanning set and basis of a vector subspace, offering opportunities to refine and develop such concepts. The use of the Excel spreadsheet was also one of the intended features. The modelling task was given after the teaching of several concepts listed under the topic of vector spaces, such as linear combination, vector space and subspace, linear dependent and independent vectors, basis of a subspace, span and spanning set, etc. Before working on the modelling task, the students had already solved mathematical questions on those concepts. At that time, the students had also worked on some modelling problems proposed in the course, involving other topics of linear algebra, but none with the use of the spreadsheet.

This adapted task, whose statement is shown in Figure 3, refers to a context of password generation by state banks in Costa Rica that was thought to be a real and known context for most of the students, as they would have the experience of using and restoring passwords as a real-world practice in numerous contexts, namely for electronic banking operations. The students were asked to draw on their mathematical knowledge to develop a procedure for creating passwords requested by a state bank.

Task 4. Bank Access Passwords

Passwords have been invented to protect our accounts, on a certain web page (e-mail, Facebook, ..., etc.), from other people who want to access our information without prior authorization. To make it more difficult for the outsider to access our account, the password must have different types of characters, such as letters, numbers, exclamation marks, etc. The number of characters will also influence the security of the password, making it more or less difficult to discover.

When someone forgets the password, a temporary password is commonly sent to the e-mail related to the account, so that the person can access the account with that temporary password, and then modify it and have a new own password again. In the case of some state banks in Costa Rica, this temporary password is made up of less than ten characters, which contains upper and lower case numbers and letters.

One way to generate these temporary passwords is by using vectors, where the entries of the vector represent characters of the desired password, the position of a specific entry of the vector being the specific position of the character in the password. In this way, a vector $\vec{v} \in IR^n$ would be a way to obtain a password of "n" characters. Thus, many passwords with the same number of characters can be generated once the vector space to which the vectors belong is defined.

- How could you develop a temporary password generator that allows obtaining passwords with at least numbers and letters of the alphabet, and with the same number of characters for all generated password?
- Develop a temporary password generator according to the bank's expectations. Explain the process used in creating your password generator.
- Using an Excel sheet, generate at least 20 passwords.

Figure 3. The task *Bank Access Passwords* translated from Spanish

The task includes questions that were likely to prompt the use of linear algebra concepts for creating a mathematical model of the real situation of password generation, in the sense that a model entails a conceptual system to describe, interpret or make predictions about a real situation (Czocher, 2018).

A reasonable mathematical model to describe the situation of the password generation would require students to choose a vector basis of a subspace of $\mathbb{R}^n$ in such a way that the spanning set would provide the mathematical representation for an infinite number of passwords. It would also be necessary to create a process of coding the generated vector coordinates and translate them into alphanumeric characters, and that would mean the transition from the mathematical representation to an object in the real world.

However, students often find it difficult to make connections between linear independence of vectors and spanning set, and there is evidence that the notion of vector basis is challenging for them (Steward & Thomas, 2010). Since the use of vector coordinates was found relevant to solve the problem, as the students could generate different passwords by using the coordinates of linear combinations of basis vectors, they were instructed to use vectors of $\mathbb{R}^n$ (the dimension should be decided) to model the problem and to resort to the spreadsheet to work on it. This would probably help students to develop a mental representation of the situation and would lead them to relate their mathematical model to the computational model in the spreadsheet, as this tool offers a convenient way to generate linear combinations of vectors through a process of creating arbitrary scalars within a certain range of values. It may also support ways of interpreting and validating the mathematical results through testing the model and checking if it satisfies the conditions of the problem.

Participants

The participants of the study were 21 undergraduate students (13 males and 8 females) that attended the linear algebra course, most of them in their first year of higher education. They had different levels of proficiency in Excel, including cases of students who had never used it. The participants were introduced to the modelling process through a preparatory lesson, where a modelling problem was proposed and solved collectively in the class with the teacher's guidance. After that lesson, and before the class work on the modelling tasks, they were encouraged to solve two other modelling problems as homework, both concerning linear equations.

The researcher (first author) assumed the role of teacher during the 70-minutes classes where the students worked on the modelling tasks. The classes followed a three-phase plan: teacher presentation of the task to the students, generally explaining the real context of the task and organizing the work in groups; students autonomous group work on the task in a computer laboratory (10 groups of 2 or 3 members), including moments of discussion between the students and between them and the teacher; and a final collective discussion, steered by the teacher, where students shared and discussed their models.

Data collection and analysis

Data collection included participant observation, with audio recording of the students' active discussions within their groups, and students' digital Excel files and written work on the proposed task.

The descriptive and interpretative data analysis (Cohen, Manion & Morrinson, 2007) draws on the extended modelling cycle with a technological component (Siller & Greefrath, 2010), by taking a cognitive perspective and aiming at capturing the students' modelling routes and their modelling sub-competencies in solving the proposed task. In the results section, we characterize and discuss the students' modelling routes, particularly performed by two groups of students, the modelling sub-competencies they showed in the transitions between the different phases of the modelling cycle, how they used the technology, namely the Excel spreadsheet, and the advantage they took from it in their modelling process. Such transitions are illustrated by the numbered arrows in Figures 4 and 6, considering the modelling cycle proposed by Blum and Leiß, 2007; Greefrath, 2011; and Siller and Greefrath (2010). Some excerpts of their work are presented to support that analysis.

Results

Students' modelling routes

The students' work on the modelling task "Bank Access Passwords" revealed different scopes of the modelling processes carried out, some of them being complete and others only partial modelling cycles. There were those who ended up in the real situation with the presentation of final results (5 groups); those who got to the real results phase and made their validation (1 group); those who finished in the phase of the real results but without validation (3 groups); and those who achieved the phase of obtaining mathematical results, without evidence of an interpretation of those results (1 group). In Table 2, we present the groups divided by those categories, along with the linear algebra concepts involved in their solutions, and their use of technology in the task.

Table 2. Description of the students' modelling routes on the task "Bank Access Passwords"

Group	Type of route	Learning of concepts involved	Technology use
1	Non-linear Complete	Vector, spanning set, basis, linear combination	Generating several real results with ten characters (letters and numbers, alternated).
2	Linear Incomplete. Final modelling step: Real results	Vector, spanning set, linear combination	No evidence of use.

3	Non-linear Complete	Vector, spanning set, linear combination	Applying and exploring concepts; programming. Generating several vectors in $\mathbb{R}^5$ (linear combinations) as an intermediate result between the mathematical model and real results.
4	Non-linear Complete	Vector, spanning set, linear combination	Generating several random numbers with six digits as mathematical results, which were then translated into real results.
5	Linear Incomplete. Final modelling step: Mathematical results	Vector	Generating several random numbers (4-6 digits) as mathematical results, without being interpreted as real results.
6	Non-linear Incomplete. Final modelling step: Real results	Vector, spanning set, basis, linear combination	Applying and exploring concepts; programming. Generating several vectors in $\mathbb{R}^5$ (linear combinations) as an intermediate result between the mathematical model and real results.
7	Non-linear Incomplete. Final modelling step: Real results	Vector, spanning set, linear combination, span	Generating several real results with six characters (alternating numbers and letters).
8	Non-linear Complete	Vector, linear combination	Applying and exploring concepts; programming. Generating several vectors in $\mathbb{R}^6$ (linear combinations) as an intermediate result between the mathematical model and real results (alternating numbers and letters). Generating passwords as real results; validating the model.
9	Non-linear Incomplete. Final modelling step: Real results	Vector, spanning set, linear combination, basis	Applying and exploring concepts; programming. Generating several vectors in $\mathbb{R}^2$ (linear combinations), as an intermediate result between the mathematical model and real results.
10	Non-linear Complete	Vector, spanning set, linear combination, basis, determinant	Generating several real results with three characters (numbers and letters, without considering ordered position).

Next, we highlight the cases of group 6 and group 8. For each case, attending to the research questions, we describe and analyse their types of modelling routes and the modelling sub-competencies revealed in solving the task, with a particular focus on the use of technology and the advantages they have taken from it in the modelling process. In any of the two groups, the modelling route developed is non-linear, including transitions to and from the technology world. Moreover, each of those groups of students used a spanning set with a different dimension. Finally, the analysis of the two groups allows to see consequences of using a standard and a non-standard basis of a vectorial subspace for solving the task and the effect on the modelling route created; it also provides insights on the modelling competencies that were required, and the use made of the technology.

The case of group 6

An incomplete modelling route

The group 6 performed an incomplete modelling route, whose modelling process draw on the use of a standard basis in $\mathbb{R}^5$ to generate passwords composed of five characters, in which the technology was used to accomplish several processes (Table 2). This group's modelling route is relevant because though the students did not get to the real situation phase (Figure 4), the mathematical model produced using a standard basis allowed students to easily programme the spreadsheet and create a computer model (Figure 5A-B) to generate passwords made of different types of characters, including numbers, letters, and punctuation marks (e.g., the period). This was the only group who considered the inclusion of special characters in the passwords.

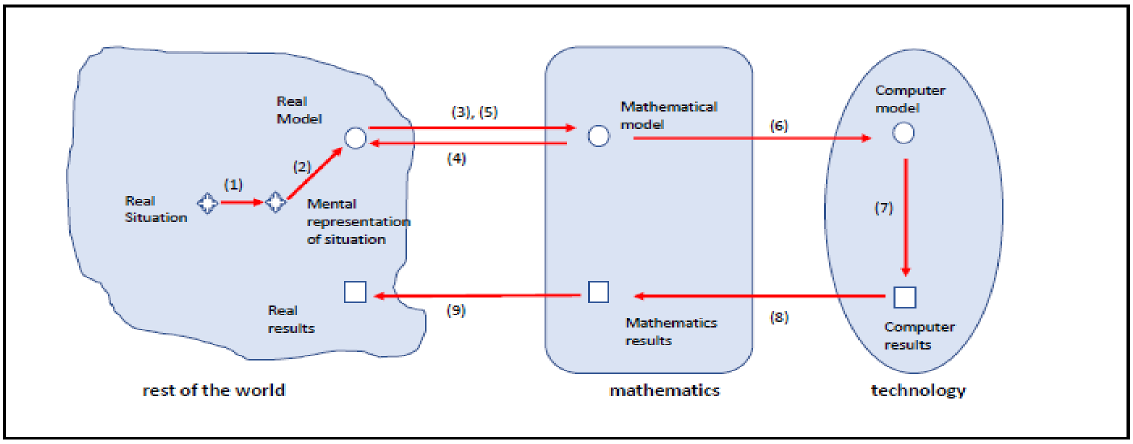


Figure 4. Modelling route performed by group 6

The modelling route is a non-linear route that shows a back-and-forth transition between the real model and the mathematical model phases. The group first decided to develop a mathematical model based on linear transformations but later turned to a model based on linear combinations of vectors in $\mathbb{R}^5$, because the students thought it would be easier to implement that mathematical procedure in Excel. In explaining their decision about abandoning the linear transformations model, the group stated:

> We thought about defining the criteria of a linear transformation, but we were not sure that the passwords could be generated [with that model]. Using a linear

> combination [as a model] was ideal because if we had the scalars and a standard basis then it would be just a matter of bounding the random values of the scalars.

Modelling sub-competencies

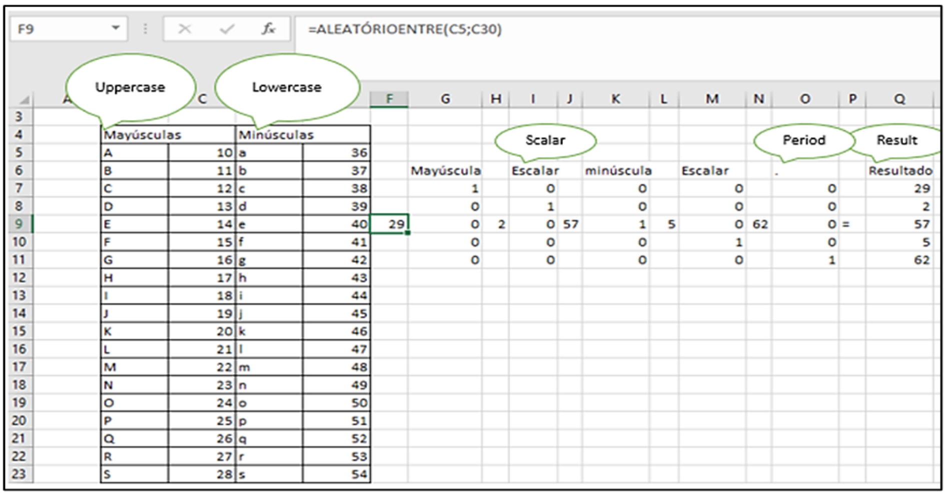
The students understood the problem of getting randomly generated passwords and looked for ways to mathematize the creation of ordered arrays of characters (structuring the task). The students mathematized the generation of those arrays of characters through the concept of linear combinations of vectors. They chose to use a standard basis of a vector space and associated the production of linear combinations with obtaining random values for the scalars (mathematizing). Their mathematical model was based on their mathematical knowledge and also on their thinking about the way it could be implemented in Excel to generate passwords. Thus, the computer model constructed by group 6 shows their competence to reduce the complexity of the task and to develop an appropriate mathematization of the real model, by generating linear combinations of the standard basis vectors (working mathematically). By implementing the model in the spreadsheet, the students achieved a mathematical method to generate a string of numbers that could be transformed by means of a coding system, thus leading to the subsequent obtaining of the real passwords (interpreting the mathematical results).

Technology use and advantages

The computer model produced by the students with the spreadsheet (Figure 5A-B) shows the creation of a coding table (left side) and a programmed algorithm to obtain linear combinations of the basis vectors in $\mathbb{R}^5$, resulting in column vectors.

The coordinates of such vectors, that is, the string of scalars used in the linear combination, represent the characters of a password (right side). Above each basis vector, the students labelled the specific type of character that the respective coefficient represented in a password (uppercase letter, lowercase letter, scalars, and period). So, for generating several linear combinations, the group used random numbers assigned to the coefficients, as it may be seen in cell F9, where the input "=RANDBETWEEN(C5;C30)" creates a random integer between 10 and 35, corresponding to the uppercase letters in the coding table. A similar reasoning was applied to cells H9, J9, L9, and N9, associated to other kinds of characters defined by the students in their mathematical model. By manually recalculating the spreadsheet, a new random number is generated for any formula that uses the RANDBETWEEN. This way, a series of vectors were generated. So, the way the students created their computer model shows their competence to work with technology in applying the concepts of vector basis, linear combination, and span.

A. Top of the Excel worksheet



B. Bottom of the Excel worksheet

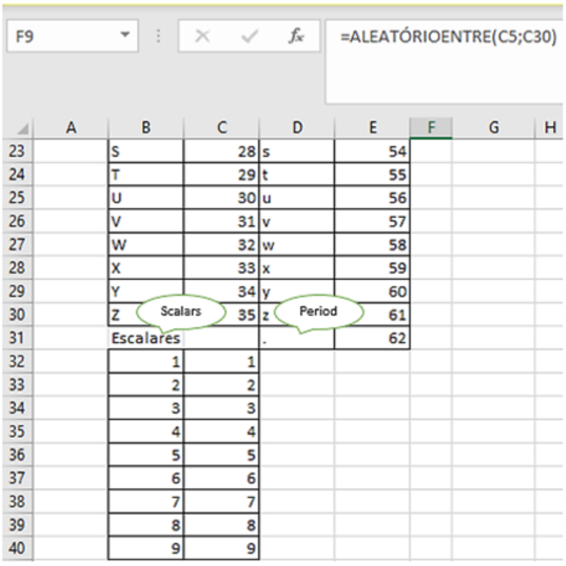


Figure 5(A-B). Computer model developed in Excel by group 6

In the case shown in Figure 5A-B, the computer result was the column vector (29, 2, 57, 5, 62) that would lead, using the coding table, to the string "T2v5." (ending with a full stop). Therefore, the computer model did not simulate the generating of real passwords, as the programming of the spreadsheet did not achieve that goal. This indicates that the mathematical model produced was not assessed in terms of its effectiveness in the real situation, a fact reflected in their modelling route where the transition from the real results to the real situation is missing (Figure 4). Although the computer model uses an algorithm to routinely create a new column vector, it did not yield the twenty real passwords required in the task.

The case of group 8

A complete modelling route

The group 8 developed a complete modelling route, by performing all the phases of the modelling cycle and returning to the real situation, but comprising different transitions between phases related to the computer model (Figure 6).

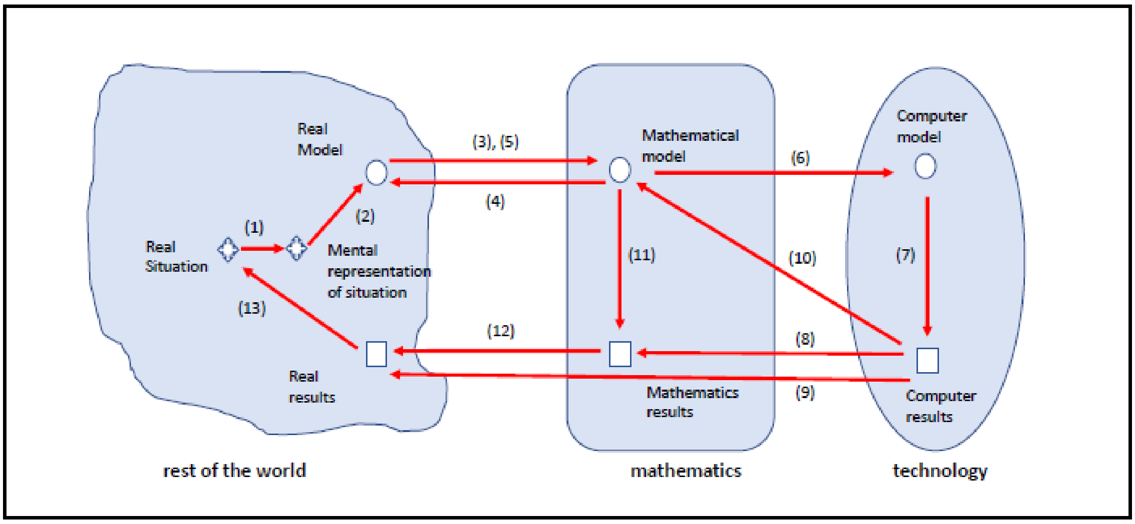


Figure 6. Modelling route performed by group 8

In contrast with other groups (e.g. group 6), the use of a non-standard basis of a subspace of $\mathbb{R}^6$ was the option of group 8. The choice of a non-standard basis composed of three linear independent vectors shows a very strong connection with their formulation of a computer model in Excel and causes several transitions in their modelling route around the computer model phase. Like group 6, this group goes through the real model twice and uses technology for working on the mathematical model, although using Excel in a more powerful way, showing advanced knowledge of the spreadsheet and achieving the real solutions required.

This modelling route shows transitions between phases that go from the rest of the world to the mathematical world, and to the technological world, being noticeable the return to phases that had already occurred.

Modelling sub-competencies

From their route, we can infer that the students showed several mathematical modelling competencies, which include the ability to reformulate their initial real model and to make use of technology to generate both mathematical results and real results. The students associated the creation of passwords with obtaining a string of numbers that would then be converted into a string of letters and digits (understanding the task). By setting 6 for the number of elements in the string, they related the strings to vectors from $\mathbb{R}^6$ (structuring the task). They considered that several different $\mathbb{R}^6$ vectors could be obtained through linear combinations of linearly independent vectors. Thus, they chose 3 vectors under these conditions, opting to use a subspace of lower dimension, to simplify the process (mathematising). Their model is based on the span of the set {(3,1,1,2,1,2), (2,1,2,1,3,2), (1,2,1,2,3,3)}. The students observed that the three vectors formed the basis of a subspace of $\mathbb{R}^6$. They represented a vector of that subspace as a linear combination of the basis vectors and defined it as the vector (x, y, z, u, v, w). To use the coordinates of the vector as the structure of a password, the students defined the following conditions:

> To the result of 'x' we will assign a lowercase letter of the alphabet according to its order; to 'y', an uppercase letter of the alphabet according to its order; to 'z', a lowercase letter of the alphabet according to its order; to 'u', 'v', and 'w', we will assign the digit in the units place of the number.

Using the spreadsheet, they defined a procedure to randomly generate the scalars of each linear combination and obtained a total of 20 vectors (working mathematically on the model). The students also took advantage of the spreadsheet to translate the vector coordinates into strings of letters and digits, which constituted the actual intended passwords (interpreting mathematical results and validating real results).

Technology use and advantages

The computer model created in the spreadsheet (Figure 7) used the basis of the subspace for generating linear combinations and for creating the real passwords associated to the generated vectors. They started by inserting the basis vectors into columns B, D, and F. Then, using the random function, obtained the scalar values in columns A, C, E, with the formula =RANDBETWEEN(0;4). The generated vector is computed in column H and the resulting alphanumeric password is found in column I.

The coding of the password vector is achieved through logical functions, namely using multiple nested IF statements. They matched each of the first three coordinates with one of the integers between 0 and 25 for determining the corresponding letter of the alphabet. They also matched each of the other three coordinates with one of the integers between 0 and 29 for determining the unit digit. In cell I8, by means of a Text function, the students applied the formula =CONCATENATE(I1;I2;I3;I4;I5;I6) under which the coded vector is translated into a string of 3 letters and 3 digits, thus generating a password in the real situation. Even though some of the alphabet letters cannot appear in any of the real passwords, as the maximum value reached by those coordinates is less than 25, the range between 0 and 4 for the scalars $\alpha_1, \alpha_2, \alpha_3$was efficiently chosen.

Students from group 8 executed the computer model in Excel, taking advantage of this technological resource to create and explicitly showing 20 real passwords (column M), by recalculating the coordinate vector defined by cells A3, C3, and E3. The outputs in the Excel sheet show that each password has six characters with the same composition: three letters at the beginning and three digits at the end, a format that satisfies the conditions established in the problem. Thus, the students revealed the competence to implement the mathematical model in Excel and work with that model to obtain mathematical and real results.

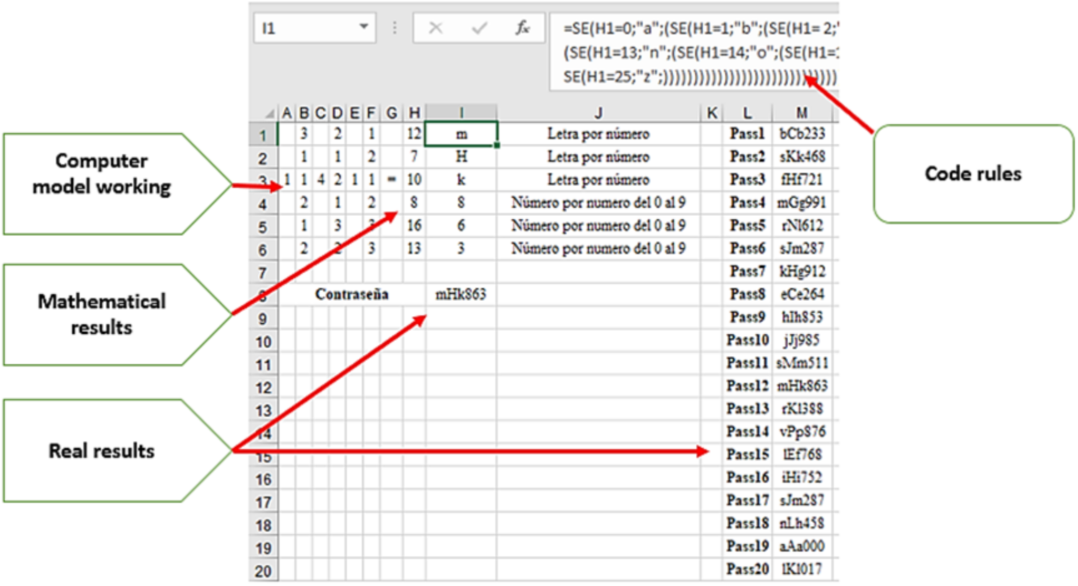


Figure 7. Computer model developed in Excel by group 8

In their report, this group explained their reasoning with the technological tool, by presenting two examples (Figure 8). Those examples show how the coding rule works for the last three components of the password vector (when the values are less than 10 or otherwise), evidencing a transition from the computer results back to the mathematical model.

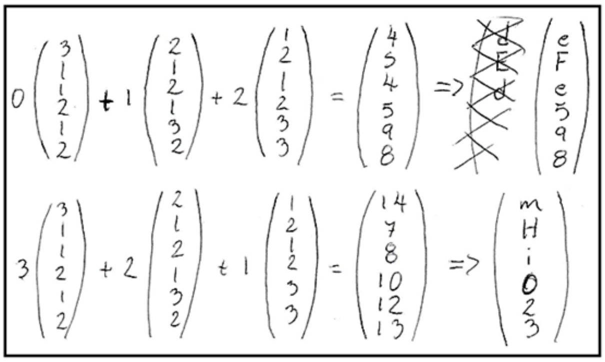
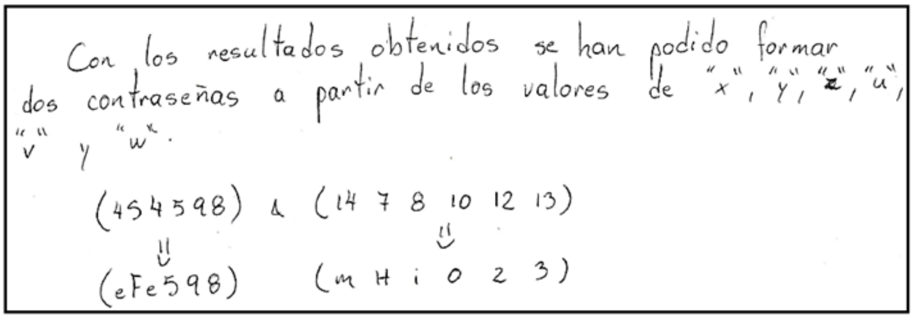


Figure 8. Examples of passwords generated by group 8

Apparently, this group of students was able to work mathematically on the model without using technology, but their computer model produced real results (real passwords), which allowed them to validate their mathematical model (Figure 9).


Con los resultados obtenidos se han podido formar dos contraseñas a partir de los valores de "x", "y", "z", "u", "v" y "w".

(454598) & (14 7 8 10 12 13)

$\Downarrow$ $\Downarrow$

(eFe598) (m H i 0 2 3)

Translation: *With the obtained results, it was possible to create two passwords from the values of "x", "y", "z", "u", "v", and "w"*

Figure 9. Real results interpreted by group 8

Conclusions

In this study, we identified different types of modelling routes performed by different groups of undergraduate students, considering the categories of complete/incomplete and linear/non-linear routes. From those routes, it was possible to grasp the modelling sub-competencies that they showed when solving the modelling task of generating bank access passwords. Their ability to use the Excel spreadsheet in the modelling process was also observed in terms of the advantages they took of the technology along those routes.

The overall results show that the students performed diverse modelling routes, namely various combinations of incomplete/complete cycles, and linear/non-linear routes. Not surprisingly, most of the observed routes were non-linear routes and some routes were incomplete. Some of the groups were only able to reach the stage of getting mathematical results or achieved real results but did not validate those results in the context of the real situation. The students who performed complete modelling routes were able to present their results in the context of the problem, moving more than once through some of the phases of the modelling cycle, including going back and forth between phases.

The students showed to be able to understand the real problem and considered a password as a string of characters, in some cases consisting only of numbers, in other cases combining letters, numbers, and even special characters, such as punctuation marks. It was also visible that the various groups were able to simplify the real situation, often assuming a certain length for the strings and making decisions about their composition. Likewise, the students identified a password with a vector of a subspace of $\mathbb{R}^n$ (the dimension of the chosen space varied across the groups), which seems to be related with the similarity between a vector and an array of numbers. Therefore, the students mathematized the real model based on vector space concepts, namely by applying the ideas of linear combination, span, and basis of a subspace. This suggests that the learning of such concepts is improved when students explore significant contexts (Wawro et al., 2012). Real-world situations provide opportunities for understanding concepts and procedures and for putting them into practice to create realistic approaches and solutions (Rensaa, 2017). The modelling task of creating bank access passwords proved to be appropriate for developing linear algebra mathematical skills (Cárcamo et al., 2016) and students' sub-competencies related to mathematical modelling.

The two specific modelling routes presented allow to capture the students' modelling sub-competencies and provide an understanding of the way their modelling processes were accomplished, namely when examining the procedures, actions and decisions made by the students in the technological world. It clearly shows that the way in which students build and run a computer model drives the modelling process in very different ways. This leads to concluding that students' competencies in using technological tools are relevant to the construction of effective models and to the creation of concrete solutions to a real problem.

The successful use of the technology was closely related to the way students made transitions between the mathematical model, the computer model, and the real model of the situation. As other researchers have argued (e.g., Greefrath, 2011), the success depends on the advantages, constraints, overload, or readiness that students identify in a technological tool for obtaining solutions in a modelling situation where mathematization is a crucial part of the process.

The computer models produced by the two groups with the spreadsheet reveal that the process of generating linear combinations was achieved by generating random values for the scalars. The established ranges were properly defined according to the real model built, namely in terms of the use of digits, letters (lowercase and uppercase), and special characters. The spreadsheet was used for producing random numbers within a given interval and for calculating; the students were able to take advantage of those affordances to quickly obtain a wide variety of numerical results (column vectors). The two groups also showed the need to incorporate an encoding process by which to transform their mathematical results into real passwords.

Using the spreadsheet to carry out the simulation of creating real passwords entailed specific processes and functions the students resorted to. One of the groups only reached part of that simulation process by creating multiple vectors in the spreadsheet and presenting a table to manually make the translation of vectors into strings. The other group achieved the simulation of password generation taking advantage of Text functions in Excel, namely the CONCATENATE function that combines two or more text strings into one. In this way, that group was able to check the validity of their mathematical model, computer model, and real model, and could also make further interpretations on the mathematics behind their results. The validation of results into the real situation is known as a sub-competence that many students lack when working on modelling tasks (Blum, 2015). Our findings show that using the spreadsheet to simulate was useful to realize the distinction between the mathematical results and the results sought in the real-world situation. It may also be hinted that taking advantage of technology to construct an efficient computer model may require several back-and-forth transitions in the modelling cycle, thus contributing to a simultaneously complete and complex modelling route.

Finally, this study suggests possibilities for further research on students' learning of linear algebra focused on the role of technology. The transitions between computer results, mathematical results and real results have shown to have a significant influence on the students' modelling processes. This justifies the need to look more deeply into the sub-competencies that undergraduate students activate when working on real-world problems, by examining possible roles of technology, such as experimenting, calculating, or simulating.

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