The nature and the aim of the examples used by a teacher in his interventions in modelling projects

A natureza e o propósito dos exemplos usados por um professor em suas intervenções em projetos de modelagem

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Abstract. This article presents the results of a study whose objective is to investigate how an experienced teacher uses examples in modelling projects, by analysing the nature and the aim of this kind of teacher intervention. Based on mathematical modelling perspectives described in mathematics education literature, in the principle of exemplarity and in a qualitative research approach, we will analyse critical episodes that took place during the practice of a mathematics teacher who, while monitoring modelling projects, used examples as a way of intervention in the students' activities. From our analysis, we could see that the examples used by the teacher in modelling projects work in different ways from those commonly used in conventional mathematics classes. As a result of the study, it was possible to outline three main natures for the examples used in this type of activity: didactical nature, socio-political nature, and fictional nature.
Keywords: modelling projects; exemplarity; teacher intervention; didactic nature; socio-political nature; fictional nature.

Resumo. Apresentamos neste artigo os resultados de um estudo cujo objetivo é investigar como um professor experiente usa exemplos em projetos de modelagem por meio da análise da natureza e do propósito desse tipo de intervenção docente. Apoiados em perspectivas de modelagem matemática descritas na literatura de pesquisa em educação matemática, no princípio da exemplaridade e em uma abordagem qualitativa de pesquisa, analisamos episódios ocorridos na prática de um professor de matemática que, durante a realização de projetos de modelagem, utilizou exemplos como um modo de intervir nas atividades dos estudantes. De nossa análise, pudemos perceber que os exemplos utilizados em projetos de modelagem funcionam de maneira diferente quando comparados

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com aqueles comumente usados em aulas convencionais de matemática. Como resultado do estudo, foi possível delinear três principais naturezas para os exemplos usados nesse tipo de atividade: natureza didática, natureza sociopolítica e natureza fictícia.
Palavras-chave: projetos de modelagem; exemplaridade; intervenção docente; natureza didática; natureza sociopolítica; natureza fictícia.

Introduction

Teachers' guidance throughout the development of mathematical modelling activities has been a topic of great research interest in the field of mathematics education (e.g., Leiβ, 2005; Leiβ & Wiegand, 2005; Lima & Araújo, 2021; Oliveira & Barbosa, 2010; Veronez & Castro, 2018). In particular, these studies are concerned with analysing the teaching performance that occurs through teacher interventions in students' activities.

In modelling activities, as pointed out by Leiβ (2005), teacher interventions may take place in different ways – such as through questions, demonstrations, etc. – and, here, we are interested in specifically discussing those that manifest themselves through the *use of examples* (Lima, 2020) as a way to help students moving forward with their modelling projects.

In a conventional mathematics class, examples are often used to illustrate a concept or a technique that has just been explained by the teacher or to show students how they can use such a technique to solve the exercises (Skovsmose, 2001). We can imagine such a situation when a teacher, after generically defining a first-degree polynomial equation and its meaning, uses examples as a way to show his/her students how they can follow and repeat a sequence of predetermined steps while solving a list of proposed exercises. Such exercises, in general, are quite similar to the example shown.

Our hypothesis, however, is that, in modelling projects, the examples used by teachers may not fulfil the same role or have the same format as those used in a conventional mathematics class, as in the case presented above. Blum (2015), for instance, offers insights about the way examples (of tasks) are crucial in elucidating different aims of mathematical modelling in mathematics education. We have confirmed this importance (Lima, 2020) when analysing the interventions of a teacher and realised that he used examples systematically in his classes while students were developing modelling projects. The teacher's extensive theoretical and practical experience with this type of projects contributed to the use of examples that were different in nature and aim, if compared to the ones commonly used in mathematics classes, and did not necessarily have an obvious and direct relationship with what students could do.

Based on our hypothesis about the different format of examples in modelling projects, we propose an interpretation of the notion of *example* through the concept of *exemplarity*

(Patronis & Spanos, 2013; Skovsmose, 1994; Vithal, 2003). This concept – from the literature in the field of education – has a close relationship with the use of examples in pedagogical practices and can offer guidance in understanding the use of examples in modelling projects. Thus, from situations which can be called *examples* in teacher's interventions and with the support of the notion of *exemplarity*, we address the following research question: how does a teacher use examples in modelling projects?

This study is based on the observation of the practice of a mathematics teacher with consistent training and extensive experience in mathematical modelling. His experience with modelling projects makes the examples he uses of a different nature and aim when compared to the usual examples in mathematics classes.

We begin by presenting the theoretical framework in the two following sections: the first regarding mathematical modelling in mathematics education and the second focused on the concept of exemplarity. After that, the methodological aspects that guided our research are described, followed by the presentation and analysis of data. Finally, we present our final remarks.

Mathematical modelling in mathematics education

In general, modelling in mathematics education can be understood as an activity in which students are involved in the search for solutions to real-life problems through mathematics. From this generic concept, modelling can undertake specific designs depending, for instance, on the intentions of the teachers who develop such practices in their classrooms or of the researchers who use modelling as a theoretical framework for their studies.

Modelling in mathematics education developed in Brazil independently of the movement that took place around the world (Barbosa, 2007). This autonomous progress, according to Araújo (2010), made it possible for Brazilian modelling to be strongly influenced by a series of socio-cultural studies that are part of the Brazilian tradition in educational research (D'Ambrosio, 1999; Freire, 1998, 2005). Thus, in Brazil, modelling has acquired singular contours, giving rise to pedagogical practices that, in addition to having educational concerns about students learning of mathematics, are characterized by a special attention to the role played by mathematics in society (Araújo, 2007, 2009; Barbosa, 2003, 2006). Such characteristics were later classified by Kaiser and Sriraman (2006) as belonging to the *socio-critical perspective* of mathematical modelling.

> It refers to socio-cultural dimensions of mathematics, which are closely associated with ethno-mathematics […]. This perspective emphasizes the role of mathematics in society and claims the necessity to support critical thinking about the role of mathematics in society, about the role of and nature of mathematical models and the function of mathematical modelling in society. (Kaiser & Sriraman, 2006, p. 306)

Although the socio-critical perspective is part of the Brazilian tradition in modelling (Araújo, 2010), practices guided by the *educational perspective* (Kaiser & Sriraman, 2006) are more common in classrooms in Brazil (Prane et al., 2016). Within this perspective, activities are mainly focused on promoting modelling practices to meet pedagogical objectives related to mathematical content learning. Kaiser and Sriraman (2006) further subdivide it into *didactic modelling* – characterized by the objectives of structuring learning processes – and *conceptual modelling* – focused on the introduction and development of mathematical concepts.

Regardless of the perspective adopted, the development of mathematical modelling activities can be configured as a possibility of strengthening the debate on issues related to *reality* and its relationship with *mathematics*, since this practice presupposes the investigation of real-life problems through mathematics; and this *reality* can be designated by different labels such as the students' daily lives, real world, real-life situations or problems, etc. (Blum & Niss, 1991). The discussion regarding the importance of using real problems to stimulate students' learning has been a topic of interest to the community (Brady et al., 2015; Conceição & Rodrigues, 2015) as well as to the field of modelling, as shown in the studies by Araújo (2002), Anastácio (2010), Dalla Vecchia and Maltempi (2012), and Rocha (2015).

For instance, Araújo (2002) describes the conflict between aspects of reality and mathematics when students are involved in modelling activities. The author describes the plan followed by some students who, contrary to what was expected, invented a situation and data for the development of the task. This fact draws our attention to the importance of in-depth discussion of the perception students have about reality when they develop modelling activities. Rocha (2015, p. 8) expands this debate by highlighting that "mathematics or mathematical models can be changed to confirm a certain hypothesis about a real situation"; in this sense, what is observed is "an inversion of roles, compared to conventional mathematics classes, in which real-life situations are modified so that the mathematics used is the same as that of routine exercises".

Specifically in relation to information and communication technology, Dalla Vecchia and Maltempi (2012) support the thesis that, in modelling environments, the reality of the cyber world works as a dimension of reality itself, and that they are distinct in the singular characteristics attributed to time and space in each scope. For the authors,

> the combination of the characteristics inherent to space and time can configure an environment that is different from day-to-day life, where it is feasible to discuss an adjective perspective of reality, encompassing imagined, projected, and possible realities. [...] Mathematical modelling, linked to this context, can assume different views and perspectives, influencing the way in which certain situations are understood and how mathematics can help the discussions that involve that space or how the situations regarding that space influence the process of teaching and learning mathematics. (Dalla Vecchia, & Maltempi, 2012, p. 988)

By following the multiplicity of perspectives, it is also possible to find, in the literature, a diversity of ways to develop and organize modelling practices. Antonius et al. (2007) state that a modelling activity can have specific characteristics depending on the time available to carry it out and the complexity of the problem studied. *Tasks in mathematics and applications*, for example, are less complex and short-lived activities that are usually carried out in a single class. *Investigations*, on the other hand, display a medium level of complexity and can last for a few weeks. Closer to the Brazilian tradition in modelling (Araújo, 2010), Antonius et al. (2007) also describe *projects*, which are characterized by their approach to broader and more complex problems, strongly linked to reality and which are naturally developed over several months. For this reason, in modelling projects, in general, students are afforded additional responsibility and autonomy in the learning process, while the teacher is seen as an advisor.

The data that will be discussed in this article were collected in classes where the activity was structured around modelling projects. It is important to emphasize that such a choice imparts the activity with a considerable degree of unpredictability, as it depends heavily on the paths taken by the students throughout the process. In this sense, we agree with the statement of Borromeo Ferri (2007) who, when pointing to the unpredictability of the development of modelling activities, highlights that, when participating in projects of this nature, students have their own *modelling routes*.

In the same way, it is also difficult to predict the teachers' interventions in modelling projects, as stated by Lima (2020). Such unpredictability can be seen through the diversity of pedagogical strategies used by them in these practices as shown by the works of Leiβ (2005), Leiβ and Wiegand (2005), Lima and Araújo (2021), Oliveira and Barbosa (2010), and Veronez and Castro (2018).

Leiβ (2005), for instance, highlights that the teacher intervention may occur in different phases of the modelling project, at different points in time, on different levels, and with different degrees of directness of his/her prompts. He also points out that the intervention can manifest itself through a variety of methods. In this sense, observing the teacher performance in modelling activities, Lima (2020) points out that the use of examples can play a very relevant role among the ways in which a teacher can intervene in student's activities.

In this article, we will focus our analysis on a way of intervention that teachers may use when developing modelling activities: the use of examples. More specifically, we want to investigate how a teacher uses examples in modelling projects by analysing the nature and the aim of this kind of teacher intervention. To support our discussion, in the following we present our understanding about the concept of *exemplarity* that will guide our data analysis.

Exemplarity

Etymologically, the noun *example* is related, among others, to the Latin word *exemplum*, which also means *model*. It does not necessarily have the same meaning as the word *model* in mathematical modelling, in which it designates a mathematical representation of a part of the real world (Niss et al., 2007). According to the Online Etymology Dictionary[1], an example is a representative of a set or a part of a whole, taken to illustrate this set or this whole. Another meaning brings it closer to the word *model*: "a pattern or model, as of something to be imitated or avoided". Another definition points to mathematics, as an instance illustrating a rule or method.

Derived from the noun *example*, in the same dictionary, the adjective *exemplary* designates, "a person or thing to be copied or imitated; model", which draws attention to this person or thing, as it is worthy of being imitated or rather avoided, depending on the type of exemplarity it represents. Whether as an element illustrating the whole or as something that must be imitated, the concept of exemplarity has been developed and used in education as well as in mathematics education (Blomhøj & Kjeldsen, 2009; Vithal, 2003).

In a theoretical discussion about exemplarity, Korsgaard (2019) claimed that, as in many other fields of investigation, exemplification plays an important role in education. "The example […] functions as a way of giving epistemic access or, in other words, making something visible or accessible to the examiner, scientist or the student" (p. 273). Korsgaard (2019) defined two dimensions of exemplarity in education: pedagogical exemplarity and didactical exemplarity. Pedagogical exemplarity refers to exemplary people, both in a positive and negative sense. It would be the case of teachers, for example, whose attitudes, knowledge, ways of communicating are inspirational or, in the opposite sense, examples to be avoided by students. The didactical exemplarity, in turn, points to didactical moments or activities that illustrate, in a representative way, some content or field of knowledge within educational processes.

In this article, we are interested in the didactic dimension of exemplarity, as we are dealing with the nature and the aim of examples used by a teacher in his interventions in modelling projects in mathematics education. To address this dimension, Korsgaard (2019) relies mainly on the work of the German educator Martin Wagenschein, from the field of science education. The latter criticizes the traditional teaching system in which mathematics and physics are taught according to the logic of the scientific structure of those fields, starting from simpler elements and moving towards further complexity. According to Korsgaard (2019), Wagenschein proposes starting from examples, specific tasks or experiments, that are more elaborated, illustrate the complexity of the content and entice students to immerse themselves in that field. In short, he proposes the use of tasks that are exemplary.

> The main idea here is not that examples can function as stepping stones or parts of a progression, but rather that on one hand they are a way of opening the subject matter, so that it becomes accessible at all, and on the other hand that these examples carry within them the whole of the subject. (Korsgaard, 2019, p. 279)

In principle, this concept of exemplarity is appropriate to our purposes in our study and, in practical terms, will be important to guide our analyses. However, Patronis and Spanos (2013) have warned about the concept of exemplarity founded on philosophical romanticism, which includes the one proposed by Martin Wagenschein. Those authors question the possibility that a harmonious whole exists and that it is represented through a particular example. To explore the nature of exemplary themes in teaching and learning of advanced mathematics, Patronis and Spanos (2013) seek to develop a hermeneutic concept of exemplarity, as they understand that reading and writing scientific texts and, therefore, teaching science and mathematics, are interpretative activities. They also rely on particular examples to represent general cases, but "focus on a more radical way of linking the 'particular' with the 'general': [they] speak of 'the particular' as critically reflecting the complexity of 'the general'" (Patronis & Spanos, 2013, p. 1994, emphasis in the original).

As Patronis and Spanos (2013) state, Skovsmose (1994) also conducted a reformulation of the concept of exemplarity proposed by Martin Wagenschein. These two reformulations, however different, are very much in tune, as Skovsmose (1994) also did not refer to a harmonious whole, but to critical and undefined aspects of society.

The principle of exemplarity, as discussed by Skovsmose (1994), was used as inspiration for the development of project work and thematic approach, moving towards approximations to the ideas of critical mathematics education. Such approaches show similarities to modelling practices according to the socio-critical perspective or even to the educational perspective conducted in Brazil. For that author, the main advantage of relying on the principle of exemplarity is the possibility of starting from concrete educational practices and expanding the discussions and reflections towards a philosophy of critical mathematics education. Therefore, Skovsmose (1994) relies himself on examples to build the foundations of the philosophy he is proposing.

According to the author, the principle of exemplarity states that the epistemic subject, through practices that thematize specific socio-political events, can understand the complexity of society and imagine different possible settings, in which their conditions could be transformed. In this way, students – the epistemic subjects – would participate in activities involving reflections about a particular social and political event which, in turn, reflect a political whole, which is not defined, is controversial and relates to critical aspects of society. To discuss the possibility of imagining different settings, Skovsmose (1994) drew on the ideas of Oskar Negt, a German philosopher and sociologist, interested in vocational education, who proposed the concept of sociological imagination: it "is oriented towards

what actual social reality can be changed into, and education acquires a political perspective" (Skovsmose, 1994, p. 77).

Based on this theoretical framework, Skovsmose (1994) describes the development of project work as examples in which students are introduced to broader social issues. Examples, in these cases, are not used as small parts to build a whole, as Martin Wagenschein criticized, but as means of making the whole visible and accessible to students (Korsgaard, 2019). The examples are specific socio-political events which reflect the complexity of critical and undefined aspects of society. This way of understanding *example* in the field of education, theoretically elaborated in this section, is supported by the literature of this field and, in particular, by that of mathematics education. From the next section, we will incorporate empirical elements to this understanding, when analysing examples used by a teacher in his interventions in modelling projects.

For Skovsmose (1994), project work, problem orientation and thematization are ways to organize school practices which are in line with the principle of exemplarity. Such ways are very similar to modelling practices, in which students choose themes from their daily life or real-life situations to be addressed through mathematics that can also become examples of more complex social situations. In our study, these types of practices produced the data collected, which will be described in the following.

Methodological approach, context, and participants

This work follows a qualitative research approach (Denzin & Lincoln, 1994) which, among other things, is focused on studying phenomena in their natural environment, seeking to develop a possible interpretation for them in terms of the meanings they are assigned by the participants of the study.

The data analysed in this article were produced during the mathematics classes of a teacher called Henrique[2], who works at a vocational high school, located in the city of Belo Horizonte, in Brazil. In those classes, some modelling projects were developed (Antonius et al., 2007) in which the students actively participated in the entire modelling process (the choice of a real-life theme that would be addressed in the project, data collection, production of the mathematical model, etc.). Henrique played the role of an advisor during the project, which was developed between the months of April and July 2018.

Henrique is a mathematics teacher with consistent training and extensive experience in mathematical modelling. His experience in the development of this type of activity is reflected in his academic production during his graduate studies: modelling in mathematics education was the object of Henrique's master's thesis and doctoral dissertation. Therefore, the episodes selected for analysis in this investigation were collected from the classes of a teacher that has significant theoretical and practical knowledge of modelling.

At the time of data production, Henrique's students were teenagers aged 14 to 17 years who attended the 1st or 2nd year of Vocational High School, enrolled in Mechanics and Environmental Sciences courses, respectively. When asked about previous experiences with modelling, all students stated that they had never participated in activities of this kind. Except for cases of students who had already participated in something similar, but did not recognize the practice by the name, all students would be doing modelling projects for the first time in their school life. This fact greatly influenced Henrique's work.

The method used to produce the data was participant observation (Adler & Adler, 1994) in the classes where modelling projects took place. The observations were recorded through field notes, audio, and video recordings of the interactions between the students and the teacher. Subsequently, the recordings were transcribed following the guidelines proposed by Powell et al. (2003).

The exploration of the data set was done through a careful search of what Powell et al. (2003) call *critical events*, which are "significant contrasting moments [...] that either confirm or disaffirm research hypotheses; [...] they may be any event that is somehow significant to a study's research agenda" (p. 417). In this sense, based on the understandings about the concept of exemplarity discussed in the previous section, we searched for all the teacher's interventions that could be considered as examples. They were identified as moments when the teacher tried to make something visible or accessible to the students (Korsgaard, 2019), as a way to create a link between particular cases and general ideas, so that one can critically reflect the complexity of the other (Patronis & Spanos, 2013).

So, the critical events were short excerpts from the videos that were exemplary and that, in some way, were related to our research question – about the use of examples in modelling projects – and to our hypothesis, presented in the introduction of this paper, that examples may work differently in modelling projects when compared to those used in traditional classes. Specifically, these critical events were moments when the teacher tried to show students how they could proceed at different phases of the development of modelling projects. Such examples should illustrate the characteristic complexity of mathematical modelling while enticing students to get involved with the activity, thus constituting themselves as exemplary representatives (Skovsmose, 1994) of modelling projects.

From the exploration of data set, we identified eleven critical events, that is, eleven moments when the teacher used examples during the development of the modelling projects by the students. The critical events were analysed through an inductive process (Creswell, 2007), in which we sought to observe any patterns and themes and, from this analysis, we have created three categories "from the 'bottom-up', by organizing the data into increasingly more abstract units of information." (p. 38, emphasis in the original). The three categories (examples used by the teacher may be i) of didactic nature, ii) of socio-political nature, or iii) of fictional nature) are presented and discussed in detail in the next section.

Therefore, none of the three categories had been determined a priori: we examined the eleven critical events by making comparisons, looking for similarities and differences. Due to space constraints in this article, not all eleven critical events are presented. Instead, we chose three of them – which, in addition to illustrating and representing the selected set of critical events, also present a considerable amount of information that will help us to highlight the analysis conducted – and elaborated an episode from each one. Episodes (Araújo, 2002) are transcriptions of short excerpts from the videos, in which data were recorded, accompanied by an explanatory text about what was happening at that moment of the development of the modelling projects. To us, the three episodes are exemplary of the analyses conducted.

Supported by the theoretical framework, in the next section we present and discuss the three categories constructed from the analysis of the eleven critical events, which describe the nature and the aim of the examples used by the teacher during his interventions in the development of modelling projects. Shortly thereafter, we present the three episodes we chose to illustrate the three categories.

The nature and the aim of examples used by the teacher in modelling projects

In modelling projects, the teacher may use *Didactic Nature* (DN) examples. The teacher uses such examples to explain how students can proceed in the development of modelling projects and, therefore, they belong to the didactic dimension of exemplarity (Korsgaard, 2019). As discussed in the introduction to this article, we hypothesize that such examples are quite different from those normally used in mathematics classes and, based on the analysis of the episodes that we will present in the next section, we could see that these examples are teacher's intervention intended to hint to the students on how to proceed in the modelling projects. They are directly related to the search for a meaning of modelling practices. In this case, we see a link with what Skovsmose (1994) sought to do for critical mathematics education, as, for him, "the examples elucidate the complexity of educational practice" (p. 59). Due to these characteristics, a relationship can be established between the examples that are didactic in nature and the didactic dimension of the educational perspective of mathematical modelling (Kaiser & Sriraman, 2006).

The teacher may also use *Socio-Political Nature* (SPN) examples in modelling projects. These examples are directly related to the principle of exemplarity, as this principle states that "it is possible to understand social complexity by concentrating on a particular event" (Skovsmose, 1994, p. 77). From examples of this nature, students can understand critical situations in society, since mathematical modelling "structures and creates a piece of reality, dependent on knowledge, intentions and interest of the problem solver" (Blum & Niss, 1991, p. 39). Therefore, examples that are socio-political in nature fulfil one of the fundamental

purposes of modelling: to elaborate understandings of day-to-day situations, of reality. Similar to the previous case, there is a strong relationship between the examples that are socio-political in nature and the socio-political perspective of mathematical modelling (Kaiser & Sriraman, 2006).

Finally, in modelling projects the teacher may use *Fictional Nature* (FN) examples. Examples of this nature have some relations with the concept of sociological imagination, which "is oriented towards what actual social reality can be changed into" (Skovsmose, 1994, p. 77). In our case, the examples used by the teacher give other dimensions to reality (Dalla Vecchia & Maltempi, 2012). They might be merely related to entertainment, or they can establish a smooth transition between the student's interests and critical issues in society, as we will show later. In this case, however, it is possible to see a relationship of the examples that are fictional in nature with more than one perspective of mathematical modelling, for example, the conceptual dimension of the educational perspective and the socio-political perspective of modelling (Kaiser & Sriraman, 2006), as we will see in the episodes.

It is important to remember that these three categories that describe the nature of the examples used by the teacher, in his interventions during the development of the modelling projects, emerged from the analysis of the eleven critical events, as we described in the previous section. Next, we present the three episodes we chose to illustrate the three categories, which will clarify the analyses and the choices we made.

Three episodes to illustrate how a teacher uses examples in modelling projects

To initiate the modelling projects, Henrique talked to students about which themes would be interesting to address through mathematical ideas and strategies. After the initial survey, Henrique asked students to form groups and agree on the real-life themes to be investigated by them. Each group then produced a written work – intended to gather information on the chosen theme – which served as framework for the subsequent activity. As the practice progressed, the teacher instructed the students about the formulation of a problem to be investigated, about the mathematization process and the production of a mathematical model to address the problem. At the end, each group would present a report containing the results of their investigations.

Below, each episode will be situated within a phase of the modelling project, through a descriptive text, followed by our interpretation of the nature and the aim of each example used by the teacher in the development of the projects.

Episode 1: *Thing* vs. *Hulk*

This episode occurred when the themes for the modelling projects were being chosen by 2nd year students in Environmental Sciences. At the time, Henrique was trying to motivate students to be more enthusiastically involved in the task. For that, he summarized the intended purpose of a modelling activity and emphasized that such an activity was different from those in the textbook, with which students were well acquainted.

> Henrique: First of all, what I want to do here is to kindle your interest in whatever theme you want. [...] The work… there will be some parts, which I'll explain as we go... But the gist of the work is that you're going to associate the use of mathematics to your life (or life in society) [inaudible] and relate mathematics to contexts that you like. [...] You will see that it is not the same as: ahh, elaborate a problem like the one in the textbook with a cute little solution and formula... You must interpret, and justify...

In this excerpt, Henrique makes it clear that modelling projects are quite different from the usual activities developed in traditional classrooms. In this sense, examples in modelling projects may be also different, as we initially hypothesized. Henrique stated that his expectation regarding the students' performance was quite different from what they had been doing so far – the idea was to value the process of interpretation and reflection of all the information they could gather, as well as the justification of possible ideas and conjectures developed during the modelling project. This initial conversation significantly aroused the curiosity of the students, enticing them to get involved (Skovsmose, 1994), as these types of activities were not very common within their day-to-day school routine. Faced with the students' lack of experience with modelling, the teacher decided to describe other activities which he had previously conducted with other students.

> Henrique: An example of a project is the following: a group wanted to find out – I don't like to say it, I do not want to influence you, but it's just an example – who would win a fight between the *Thing* and the *Hulk*[3]? […] At the time, they wanted to create this confrontation and began to use measurements of those guys characteristics. [...] In the end, [with] the mathematical criteria they had created, they came to two paradoxical conclusions: one is that the *Hulk* has infinite strength to maul the *Thing* for the rest of his life and the other is that the *Thing* has infinite resilience to take the beating for the rest of their lives and neither would win the fight.
>
> Luan: But I don't understand the relationship of mathematics with this stuff.
>
> Henrique: So, to reach that conclusion, [...] they started using the formula for force in physics, formulas involving time... and they obviously used their creativity, which is the main thing you should do. You should create! You are not going to use something that already exists, ok? [...] One of the things the boys did at the time was trying to estimate the energy, in Joules, with the physics formula, of one

Genki Dama[4]. For instance, how many atomic bombs is one *Genki Dama* equivalent to?

Lorena: Aah!

In this episode, Henrique's intention was to illustrate, through an example of a modelling project developed by his former students, how the groups could proceed in the development of their respective projects, by giving, therefore, a DN example. The example presented, however, is of a modelling project whose theme is part of fictional reality, of comic books, which makes the example also a FN example. Because the example is far-off from what students know from math classes, Luan questions its relationship with the school subject. Then, the teacher describes how mathematics, physics, creativity, and imagination can work together in solving a problem of fictional reality. By exemplifying with a project developed by former students, the teacher guides his current students on how they can proceed, using the school subjects and being creatives in the process. Therefore, Henrique seemed to have pedagogical and subject-related goals and the example is related to the educational perspective of modelling (Kaiser & Sriraman, 2006), in both its didactical and conceptual dimensions. So, in episode 1, the teacher used an example that is both a DN and an FN example, but not an SPN example.

Episode 2: Sports and their variables

In this episode, Henrique guided a group of 1st year students in Mechanics who had chosen sports as the theme for their modelling project. At that time, the group had already conducted their initial investigation and produced a text with all the information they considered relevant about the theme. During their conversation, the teacher conducted a descriptive assessment of what the group had already produced and gave examples of possible problems which could be addressed within that field.

Henrique: You've already taken a quantitative look at sports; two different types of quantitative approach. Within a game, the question of the score, and within a tournament, the question of scores to determine ranking. This will give us an opening to study a lot of language for mathematical functions, however, not two-variable functions, we need more variables. For instance, one cannot define the score of an American football team by the number of touchdowns, because there can be touchdowns, field goals, safeties, extra points, two-point conversions. So, there are several free variables which determine a dependent variable, which results from them all. So, I have a suggestion for you: since you are talking about several sports and, since the scores can be different in each sport, use that as context, so that we can study the language of functions with several variables [...] and this may not be only for the games, but for the whole championship as well. For example, how can the final ranking of a Formula One pilot be defined in the championship as a function of the number of wins? You cannot! As a function of the number of victories, number of second places, third places... it is one variable as a function of

several others. All in all, we can summarize it... A series of sports, creating a series of formulas and not only presenting characteristics of the sports, but the mathematical characteristics of the languages used in such formulas. Do you think this is an interesting path to follow?

Alberto: Yes.

The group had already gathered a lot of information about the theme addressed in their modelling project, and the teacher exemplified possibilities of problems and mathematical treatment of such information. He gave examples of different approaches ("within a game or within a tournament") and of how to specify the variables involved in scoring different sports (American football, Formula One). In this 1st year class, the teacher intended to use the concept of function to model situations chosen by the groups, revealing a typical goal from the educational perspective of modelling and, in this episode, he exemplifies how this could be done from the theme of sports, suggesting a connection with the realistic perspective of modelling (Kaiser & Sriraman, 2006). Thus, we can detect two concurrent natures of the example: an SPN one, when the teacher discussed the social setting of scoring and ranking in different sports and exemplified how it can be mathematically treated; and a DN one, as the teacher gave examples to guide the group on how to proceed with their project, as shown by the question "Do you think this is an interesting path to follow?". Therefore, in episode 2, the teacher used an example that is both a DN and an SPN, but not an FN example.

Episode 3: *Vibranium* and democracy

In this episode, students met with Henrique to define their problem. This conversation took place with 2nd year Environmental Sciences students. The group had chosen the theme *Vibranium*, a fictional material referred in comic books, characterized by its colossal resistance. In that reality, *Vibranium* is considered the most resistant material there is and, though scarce around the world, it is abundant in *Wakanda*, a fictional country in Africa. Based on the information collected by the group, Henrique made a few comments regarding the orientation of the project.

Henrique: Firstly, you must make the connection – you are talking about the *Marvel*[5] universe and weapons in the real world; at some point there will have to be a dialogue. We haven't had any so far... It is ok, we are still in the introduction [of the modelling project] –, but there must be a dialogue. [...] You have put here, in the introduction, *Vibranium* and weapons. This 'and', a connection must be made somewhere… How? These cannot be two parallel themes, two parallel projects. You want to put these two things together. So, I thought the following: let us pretend – and we really must – that *Vibranium* did exist and was available in the world. Obviously, this would be an imaginary scenario, but you would create, using the references from the comic, similarly to what you have done; of where it might be found, etc. What would

> the consequences of such an assumption be? Geopolitical, and military consequences... Just imagine, a correlation with what happens nowadays with oil. Where there is oil, many things happen, including people going crazy to get there, where the oil is, and bombard those places with their "form of democracy". 'I am going to liberate you because I am the voice of democracy and global good, so I am going to invade your country.' However, no one wants to invade a country such as Mali where there is a civil war, and people are dying… there is no oil there... Based on *Marvel's* references, imagine, for example, that *Vibranium*, existed in certain places, how would that affect the geopolitics of the world or of that region? You will have to conduct a simulation, maybe... It is a suggestion: as you mentioned both things concomitantly, now it is time to connect them and use mathematics to do it, maybe you will be able to calculate a military budget for the USA – as you are talking about American military weaponry – a war budget with and without *Vibranium*, and how these variables would play out. And more general consequences as well, such as where *Vibranium* would come from...

In this episode, Henrique gave examples of possible arguments ("let us pretend that *Vibranium* did exist and was available in the world" and to do "a correlation with what happens nowadays with oil") and possible questions ("where it might be found, etc. What would the consequences of such an assumption be?") to connect the two parts of the project that, until that moment, were in parallel. So, to guide the group in this episode, Henrique made a comparison between two elements: oil (a real fossil fuel) and *Vibranium* (a fictional metal), which indicates, from the start, that this is an SPN and an FN example. The observations he made about the two elements explicitly referred to global political relations based on ownership of energetic resources, which emphasizes a political and critical character both to the SPN and the FN example. In this episode, we also noticed the presence of a DN example, since the teacher called the students' attention to the conjunction *and*, in the introduction of the text produced by the group, and made suggestions for eliminating the disconnection present in their work. He also reminded the group about the necessity of using mathematics in the project. Thus, the examples used in this episode are SPN, FN, and DN examples. Therefore, they are connected with the educational and the socio-political perspectives of modelling (Kaiser & Sriraman, 2006).

Summing up

In the three episodes, the teacher made interventions (Leiβ, 2005; Lima, 2020), using examples, to guide the groups in the continuation of their respective modelling projects. Such examples did not "function as stepping stones or parts of a progression", as questioned by Korsgaard (2019, p. 279), but they suggested some possible ways that could be followed, or not. At times, the teacher's words had a strong guiding character, with excerpts such as "you can do this and that", but they were soon followed by a *for example*, to illustrate what

he had just said. Thus, "the examples elucidate the complexity of [the] educational practice" (Skovsmose, 1994, p. 59) of mathematical modelling, giving meaning to it, and, at the same time, thematizing specific socio-political or fictional events. Thus, the examples move away from those criticized by Martin Wagenschein and get closer to the notion of exemplarity, as discussed by Skovsmose (1994).

Table 1 presents a summary of the analysis carried out.

Table 1. Natures and aims of the examples used by the teacher in modelling projects

Example of	Main characteristic	Aims	Illustrated by episodes
DN	Illustration of different procedures in the development of modelling projects	To guide students on how they can proceed	1, 2, 3
SPN	Connection between particular events and general situations in reality	To guide students to elaborate understandings of social or critical situations of reality	2, 3
FN	Consideration of fictional situations or other dimensions of reality	To arouse students' interest or to establish a relationship with concrete reality	1, 3

All the three episodes feature examples that are didactic in nature, which is natural in school settings, and it is in syntony with the educational perspective of modelling (Kaiser & Sriraman, 2006). The teacher intervention was necessary (Lima, 2020) in providing examples to guide students in developing the modelling projects, as it was a school task and the students had little experience in this kind of activity. We must reiterate that the examples shown here are not a sequence of steps to be followed, but a way to assign meaning to the practice (Skovsmose, 1994) of modelling, which is exemplary to Henrique's students and can be exemplary to the field of mathematics education.

As part of the DN examples, we point out the importance of Henrique's guidance on how to conduct mathematization in modelling projects. According to Niss et al. (2007), at that moment "the relevant objects, data, relations, conditions, and assumptions from the extra-mathematical domain are then translated into mathematics, resulting in a mathematical model through which to address the identified problem" (p. 9), whether such extra-mathematical domain is real or fictional (Araújo, 2002; Dalla Vecchia & Maltempi, 2012). As discussed in Araújo and Lima (2020), the mathematization process is usually very complex for students, and teacher interventions are elemental to help them tackle difficulties.

The SPN and the FN examples displayed different features in episode 3 and, respectively, in episodes 2 and 1. The fictional situation in episode 1 and the social situation in episode 2 had a more entertaining character, or, at least, that was the tone of the exemplifications given by the teacher at the time. Conversely, in episode 3, students' interest

in that fictional reality was used by the teacher to foster their sociological imagination (Skovsmose, 1994) towards a critical and real socio-political situation, imparting a socio-critical character onto mathematical modelling (Araújo, 2007, 2009; Barbosa, 2003, 2006; Kaiser & Sriraman, 2006).

This synthesis allows us to make our final remarks, which will comprise the following section.

Final Remarks

In the present article, we proposed an investigation that seeks to answer the following research question: how does a teacher use examples in modelling projects? Thus, based on the discussions regarding exemplarity presented by authors such as Korsgaard (2019), Patronis and Spanos (2013) and Skovsmose (1994), we turned our attention to the practice of a mathematics teacher with consistent training and extensive experience in mathematical modelling who, through teacher interventions (Leiβ, 2005), used examples as one of his teaching methods (Lima, 2020), while conducting modelling projects (Antonius et al., 2007).

From the identification and analysis of eleven critical events (Powell et al., 2003), that is, eleven moments when the teacher used examples while his students were developing modelling projects, we have created three categories to describe these examples: they may be of didactical nature (DN) or of socio-political nature (SPN) or of fictional nature (FN).

Such examples are directly linked to the notion of exemplarity, regarding its characteristic of being an illustration of the modelling ideas, since this practice considers aspects related to students' interests and reality as fundamental for their conduction. From the analysis, we also realized that, in mathematical modelling projects, examples may have one or more natures simultaneously, exerting different kinds of influence, depending on the intentions of the teacher at the time he uses such pedagogical resources, as well as his knowledge regarding politics, geography, society, mathematics, physics, sports, comic books, super-heroes, in short, the general knowledge of the teacher. Therefore, whether to guide the students in the conduction of school tasks or to lead them to reflect and problematize social, political, or fictional situations, the use of examples constitutes an important teaching strategy for modelling in mathematics education.

We also point out that, although the study was developed in an environment where the teacher was experienced in modelling and the students were not, we believe that the nature and the aim of the examples, investigated in this study, may emerge in other circumstances. However, we have the hypothesis that the nature(s) of the examples used by the teacher may be related to the subjects' different experiences. Thus, it is important that further studies be dedicated to investigating the relationship between exemplarity and modelling from the standpoint of previous experiences of all subjects – teacher and students – involved in the practice.

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Notes

[1] https://www.dictionary.com/browse/example?s=t
[2] To safeguard the privacy of the participants in the research, names attributed to Henrique and his students are fictional.
[3] The *Thing* and the *Hulk* are fictional *Marvel Comics* book characters known for their extreme strength and resilience.
[4] *Genki Dama* is one of the powers of *Goku*, a fictional character from the anime *Dragon Ball Z*. The energy concentrated in this power is derived from the vital energy of several other living beings on Earth, who lend their energy so that *Goku* can amass a certain amount of power. The more living beings bestowing their power, the larger and more powerful *Genki Dama* will be.
[5] *Marvel* universe is the set of productions (comics, books, films, animated series, etc.) created and developed based on the comics produced by *Marvel Comics* Publishing.

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