Teaching and learning combinatorics in secondary school: a modelling approach based on the Anthropological Theory of the Didactic

Ensino e aprendizagem de combinatória no ensino secundário: uma abordagem de modelação baseada na Teoria Antropológica do Didático

Susana Vásquez
Universitat de Barcelona, Col.legi Natzaret
Spain
susanavasquez.elias@gmail.com

Berta Barquero
Universitat de Barcelona
Spain
bbarquero@ub.edu

Marianna Bosch
Universitat de Barcelona
Spain
marianna.bosch@ub.edu

Abstract. This paper focuses on the role of combinatorics as a modelling tool to inquire about different situations involving counting and simulation with real objects. Based on the Anthropological Theory of the Didactic, our research presents the design and implementation of a Study and Research Path (SRP) for compulsory secondary education in the area of combinatorics. The SRP starts from a generating question about discovering which padlock (among several) is safer. The empirical results correspond to a second implementation of the SRP with grade 10 students in a Catalan school with a long experience in educational innovation. We distinguish two modelling phases. First, we look at the role of combinatorics in the modelling process that emerged from the initial padlocks' problem situation. We consider students' construction of models to represent their explorations through the interaction with the padlocks, highlighting the importance of naming and defining the variables and the relationships used to characterise the types of padlocks. Second, we analyse how students simulate and validate these elementary combinatorial models before generalising them to explore other systems beyond padlocks.

Quadrante: Revista de Investigação em Educação Matemática
https://doi.org/10.48489/quadrante.23878
Received: March 2021 / Accepted: November 2021

Keywords: combinatorics; mathematical modelling; secondary school; Study and Research Paths; modelling praxeologies.

Resumo. Este artigo centra-se no papel da combinatória como ferramenta de modelação para investigar e estudar diferentes situações que envolvem contagem e simulação com objetos reais. Com base na Teoria Antropológica do Didático, a nossa investigação apresenta a conceção e implementação de um Percurso de Estudo e Pesquisa (PEP) para o ensino secundário obrigatório na área da combinatória. O PEP parte de uma questão geradora sobre a descoberta do cadeado (entre vários) que é mais seguro. Os resultados empíricos correspondem à segunda implementação do PEP com estudantes do 10.º ano de uma escola catalã com uma longa experiência de inovação educacional. Distinguimos duas fases de modelação. Primeiro, analisamos o papel da combinatória no processo de modelação que emergiu da situação problemática inicial dos cadeados. Consideramos a construção de modelos pelos alunos para representar as suas explorações através da interação com os cadeados e a importância de nomear e definir as variáveis e as relações utilizadas para caracterizar os tipos de cadeados. Em segundo lugar, analisamos a simulação e validação destes modelos combinatórios elementares utilizados pelos estudantes e a sua generalização para explorar outros sistemas para além dos cadeados.
Palavras-chave: combinatória; modelação matemática; ensino secundário; Percursos de Estudo e Pesquisa; praxeologias de modelação.

Introduction

This article focuses on teaching and learning combinatorics in secondary school from a mathematical modelling perspective. Several authors underline that combinatorics is an essential topic in the mathematics curriculum, with a rich structure of powerful principles that underlie several areas, such as counting, computation, and probability (English, 1993, 2005). According to English (2005, p. 121), combinatorics may be defined as “a set of principles of calculation involving the selection and arrangement of objects in a finite set”.

The recommendation of including combinatorics in the school mathematics curriculum has been endorsed, for several years, in international and national curriculum debates (Batanero et al., 1997; Kapur, 1970). Combinatorics has been highlighted as closely related to other curricular domains that refer to the “way of counting and computing” and “ways to build models of representation” (English, 2005). However, students often have great difficulty when addressing complex counting problems. For example, Batanero et al. (1997) propose taking a more in-depth look at students’ mistakes when solving combinatorial problems to identify the variables that might influence their difficulties. These studies are first necessary to help researchers understand the nature of students’ difficulties and the reasons behind their mistakes and, secondly, to analyse how students perform in combinatorial activities.

There are several types of problem situations involving combinatorial knowledge. As explained by English (2005), these problems on the one hand usually include the fundamental counting principle (DeGuire, 1991) requiring the use of systematic lists, tables, or tree diagrams. On the other hand, they can include combinatorial configurations (Batanero, et al., 1997; Dubois, 1984). The difficulty of the resulting combinatorial configurations depends on the type of combinatorial operations and the nature of the elements to be combined. Concerning combinatorial operations, we can distinguish between arrangements, permutations, and combinations, depending on the number of elements counted and whether the order is important. The elements to be combined are usually digits, letters, or objects, among others. However, as Lockwood (2013) underlined, the literature on combinatorics education is not very developed and has not yet addressed such ways of thinking at a level that enables researchers and educators to understand how students conceptualise counting problems.

Our study concerns a proposal to teach combinatorics at the secondary school level. We explore the role of combinatorics from a mathematical modelling approach to inquire and study different situations involving counting and simulation. We address the following research questions: *How to approach combinatorics problems from a modelling perspective? How can this modelling perspective help design, implement, and analyse a teaching proposal about combinatorics in secondary school?*

Our research is based on the Anthropological Theory of the Didactic (ATD) and its conception of mathematical modelling. The ATD approaches teaching and learning phenomena from a perspective that is at the same time epistemological (putting the knowledge to be taught and learnt at the centre of the analysis) and institutional (considering mathematics as a human activity carried out in different social settings). As explained in the following section, we use the notion of *modelling praxeologies* and the proposal of the so-called *Study and Research Paths* (SRPs) for the teaching and learning of mathematical modelling. We rely on past results about the design and implementation of SRPs (Bosch, 2018; Chevallard, 2006, 2015) and adapt them to the area of combinatorics in compulsory secondary education. We propose a SRP that starts from a generating question about how long it would take to open some particular types of padlocks. This paper focuses on two consecutive implementations of this SRP with grade 10 students in a Catalan school with a long educational innovation experience.

We will distinguish two phases in the modelling activities. First, we look at the role that combinatorics plays in the modelling process that emerged from the initial padlocks problem situation. We analyse students' construction of combinatorial models to represent what they explore through the interaction with the padlocks. From a modelling perspective, this interaction highlights the importance of naming and defining the variables at stake and the relationships used to characterise the types of padlocks. Second, we analyse the simulation and validation of these elementary combinatorial models used by the students and how they were then generalised to explore other systems beyond padlocks.

The ATD as a framework for instructional design in modelling

Despite a diversity of conceptualisations of modelling activities (Barquero, Bosch, & Wozniak, 2019; Perrenet & Zwaneveld, 2012), there is a widespread consensus about the modelling process and its decomposition in different steps synthesised in various versions of modelling cycles. Many modelling cycles can be found in the literature with varying approaches (e.g., Blum & Leiß, 2007; Borromeo Ferri, 2007; Galbraith & Stillman, 2006; Kaiser & Sriraman, 2006; Niss & Blum, 2020). They have been particularly helpful in analysing the cognitive processes students follow while solving modelling activities, studying what happens in each step of the modelling process, exploring the paths followed by students or teachers, and designing new modelling activities.

In the case of the ATD, modelling is linked to the notion of mathematical activity by assuming that doing mathematics mostly consists of producing, transforming, interpreting, and developing mathematical models (Chevallard, 1989; García et al., 2006).

On the one side, mathematical activities, as any other human activity, are described in terms of *praxeologies*, which are the primary tool proposed by the ATD to approach knowledge and activities in institutional settings (Chevallard, 1999). A praxeology is an entity formed by a combination of *praxis* – the know-how or ways of doing – and *logos* – an organised discourse about the praxis. The *praxis* block contains *types of tasks* and sets of *techniques* to carry out the tasks, while the *logos* block includes a *technology* (a discourse about the techniques) and a *theory* to justify the technology. This quartet provides a unitary vision of human activities without dissociating the "doing" from the "thinking and telling about the doing".

On the other side, the ATD proposes a broad notion of modelling to describe knowledge production (Chevallard, 1989). The two main elements are the notions of *system* and *model* that represent more a function than an entity. A model is something one considers or elaborates to gain information about a system. For instance, a fraction can be used as a model of a certain proportion of objects (playing the role of system). Still, it can also be considered a system to be modelled with another model, such as an equation or an algebraic expression. In this general conception, both models and systems can be mathematical or extra-mathematical. It depends on how they are used and how one considers them. Their role can also be exchanged. For example, let us consider the system formed by a class of 16 boys and 19 girls. We can model it with the fraction 19/35 to get knowledge about the proportion of girls in the class. Reciprocally, to establish that $19/35 < 20/36$ (an inequality that we are now considering as our system), one can use the class with boys and girls as a model and say: "if in a class with 19 girls and 35 children, one more girl arrives, the proportion of girls clearly increases". It is now the extra-mathematical model that helps us getting new knowledge about the mathematical system. This situation is very common at school, for instance when teaching negative numbers using lifts, debts, and temperatures.

As in other approaches, the modelling process includes different phases, as delimiting the system to be studied, constructing the model, working with the model to obtain information about the system, and coming back to the system to interpret, validate and extend the results obtained. It becomes a recurring process when the information brought by the model introduces new questions about the initial system, the validity of the model or the relationship between the system and the model.

We can now join the perspective about modelling with the description in terms of *praxeologies*. We start from an initial system, where some particular questions and tasks are posed; we use a technique to produce a model of the system underpinning the tasks, according to some hypotheses assumed to delimitate the system. We sustain this praxis by notions, tools, and justifications provided by the technology and the theory that justify why and how we can use the modelling techniques. This work corresponds to what we call *modelling praxeologies* (Barquero et al., 2019; Wozniak, 2012).

Moreover, Serrano et al. (2010) explain that

> the productivity of the model, that is, the fact that it produces new knowledge about the system, requires a certain 'fit' or 'adaptation' to the system. This process is rarely done at once. It requires a fourth and back movement between the model and the system, in a sort of questions-answers or trial-error dynamics. (p. 2193)

Therefore, once a given system has been modelled, new questions usually emerge. New modelling *praxeologies* can be developed by integrating the model produced into new techniques to solve new tasks within a more developed *logos*. Following García et al. (2006), we can consider modelling as a *process of reconstruction and articulation of praxeologies of increasing complexity*. Thus, modelling also appears as the construction of a sequence of mathematical *praxeologies* that become progressively broader and more complex.

In this process, modelling is a continuous and recursive process since each model (or *praxeology*) proposed can, in turn, be questioned and become a system for a new modelling process. It enables the connection and coordination of mathematical models (or mathematical *praxeologies*) into broader and more complete knowledge organisations.

This paper presents an example of a modelling project about the security of different types of padlocks, particularly concerning the time required to open each padlock. We use a didactic device proposed in the ATD to design and implement inquiry processes for educational purposes called *study and research paths* (SRPs) (Bosch, 2018; Chevallard, 2015). As we explain later, a SRP aims at providing or elaborating an answer to an open question *Q* through an inquiry process. This process (or "path") involves raising derived questions, searching already available pieces of answer or knowledge tools, mobilising knowledge, and other kinds of resources to validate, adapt, and develop the information found.

Our research methodology corresponds to the didactic engineering process (Barquero & Bosch, 2015) structured in four steps. First, the identification of didactic phenomena to address. In our case, they correspond to the formal character of current teaching of combinatorics at secondary level in Spain (Roa et al., 2003). The second step refers to the *a priori* analysis of a given teaching proposal under certain conditions: here, the design of an SRP about padlocks security as an appropriate initial system to justify and develop combinatorics as a modelling tool. The implementation of the SRP appears as the third step or *in vivo* analysis, to gather information and evidence about the implemented didactic process. Finally, the fourth step corresponds to the *a posteriori* analysis that goes back to the conditions established for the running of the SRP, its design and the didactic phenomena at stake.

In SRPs, modelling *praxeologies* appear during the inquiry to approach questions and develop answers. In the next sections, we use some of the main traits of SRPs, in particular:

- The starting point of an SRP, and consequently of the modelling process, is a generating question Q posed by the teacher and addressed to the community of study – the students and the teacher. In our case, the generating question is about inquiring into *which padlock is safer.*
- The study community addresses the generating question by opening many derived questions and proposing partial answers to these questions. An arborescence of questions and answers is used to describe the possible paths to follow (*a priori* design) or those actually covered (*in vivo* or *a posteriori* analysis). Modelling processes then appear as arborescences (or tree-structures) of questions and answers that establish possible connections among them.
- In this question-answer dialectic, mathematical modelling appears as a recursive process that includes the mathematisation of extra- and intra-mathematical systems. This process can start from an extra-mathematical system modelled using mathematical tools (arithmetic operations, figures, formulas, equations, functions, etc.). Then the mathematical models are developed to get information about the system. At that point, the models can assume the role of mathematical systems to be further modelled, and the process starts again. Sometimes, in this process, the initial system can end up representing the model used to study it, the recursiveness leading to a situation where the system models the model –now acting as a system.

In this paper, we distinguish among different modelling phases to clarify this dialectic between the system and the model(s) considered. As a general description, in the first phase, we look at the role combinatorics plays in the modelling process that emerged from the initial padlocks' problem situation. We analyse students' construction of models to represent what they explore through the interaction with the padlocks and the importance of naming and defining the variables and the relationships used to characterise the type of padlocks. This leads the students to build up a diversity of representations of combinatorial models involving laws and operations. Second, we analyse the validation of the elementary combinatorial models and their generalisation to explore other systems beyond padlocks. In this process, we will see the padlocks themselves acting as models of the new systems to study. During these two phases, we describe the modelling *praxeologies* that emerge and their evolution towards more complete situations allowing students to use the typology of padlocks to address new combinatorial problems.

Design of an SRP about combinatorics for secondary school

Institutional context and conditions for the implementation

The first experimentation of the SRP happened in April-May 2020. After a process of analysis and improvement, this same SRP was carried out again in February 2021. In both cases, there were two groups of grade 10 students, from 14 to 16 years old, at Col·legi Natzaret, in Esplugues de Llobregat, a town near Barcelona. In this paper, we focus on the second implementation.

The SRP methodology was new for the students. However, Col·legi Natzaret is a school with a long tradition in student-centred pedagogy. Since grade 1, students have been used to different kinds of innovative instructional proposals, like project-based learning, cooperative work, strategies to develop metacognitive skills, and the use of digital tools.

According to the teachers, in the academic year 2020-2021, grade 10 students were relatively homogeneous with good grades; they were used to work in cooperative teams and had strong autonomous working skills, predisposition and motivation. For these reasons, teachers concur that classroom management was not, in general, a constraint. Regarding their prior knowledge, these students had only done one activity related to combinatorics in the previous year. In this activity, they were confronted with a simple counting case, where it was feasible to manually count the total number of combinations.

In the second implementation of the SRP about padlocks, there were two groups of 30 students and two teachers (one per group), both with experience in teaching, one being also a researcher in didactics and first author of the paper. The teacher-researcher led the design of the SRP together with the research team. All decisions were there discussed by both teachers, and they plan together the details of the implementation. They decided to organise the students working teams beforehand following a “level” criterion: students were sorted by their last term marks and clustered into teams of 6 students. Therefore, both groups had five working teams of six members.

According to the course’s current pedagogical organisation and the assessment of competences promoted by the official curriculum guidelines, the weight of the working team’s tasks was 70%, leaving the other 30% to individual work. Working team tasks assessment included a workbook, daily reports, and an oral exposition. Individual work was assessed with a final test and the students' contributions and participation in classroom discussions. The material prepared for the SRP included: a guided digital workbook for the students to show their research about the padlocks; diary templates to be filled by the teams to record the list of questions addressed, answers found, and new questions arisen; a summary about the different combinatorics formulas; a list of problems related to the combinatorial unit; and an online survey. In the survey, students had to give their opinion about: the calendar organisation; the difficulty and the amount of work; the importance of the padlocks in their understanding of the mathematical knowledge; the importance of the *masterclass*[1]; whether they were willing to repeat a similar unit organisation about another topic.

Teachers decided to meet regularly after each session to discuss what happened and make decisions about the next sessions. When needed, they included the research team in the discussion. They also recorded their vision of each session in a shared logbook.

A priori design of the SRP about padlocks

The first design of the SRP started from the generating question about: *How long would it take to open some padlocks?* Five different padlocks were used to introduce this question and, quickly, students engaged in describing the number of possible combinations to open each kind of padlock. In the second implementation, the formulation of the generating question was broader: *Which padlock is safer?* This second formulation aimed to let students decide by themselves to look at how many combinations each padlock has, to compare each padlock's safety. Moreover, we also expected that students would look for other characteristics of the padlocks and inquire about their physical traits and strengths.

The initial question was introduced from five initial padlocks (Figure 1), each of them with different properties:

#1 #2 #3 #4 #5

Figure 1. First set of padlocks used by students

Padlocks number 1, 3 and 4 have similar operability, the digits of the combinations can be introduced by rotating some wheels. To begin with, padlock number 1 allows any combination of 4 digits, in other words, we can introduce any number from 0000 to 9999, so there are 10.000 possible combinations. Padlock 3 works in the same way: we can introduce any date from 00-JAN-00 TO 39-DEC-99 (the padlock doesn't recognise if the date is real or not), so there are 40·12·100 possible combinations. Finally, padlock 4 has the same mechanism as padlock number 1 but with 5 cells. We can introduce 10 different letters in any cell, which gives 100.000 possible combinations because the padlock does not recognise if the word has any meaning.

Padlock number 2 is different than the previous ones. It does not have wheels, but buttons. It opens when the correct three buttons are pushed. When a number is pressed, it remains activated until it is pushed from the back side of the padlock. Therefore, in this case, numbers cannot be repeated and the order of activation of the buttons does not influence the final combination. Then, this padlock has $\binom{10}{3}$ different combinations.

Finally, padlock number 5 is equipped with a dial that allows a choice of four possible directions (up, down, left and right). The correct password is a combination of four of these directions, enabling repetitions. This padlock is actually very similar to padlocks #1, #3 and #4, but with a dial instead of wheels. It has $4^4 = 256$ possible combinations.

In a first phase, students were expected to address the questions Q_1: *How many combinations does each padlock have? Which strategies can be used to count them?* They were expected to use some techniques to understand the system. In particular (1) listing the possible codes or combinations for each padlock (for instance, writing them manually or using *Excel*); (2) making an initial list with the exemplification of possible codes, and using arithmetic calculations to facilitate the total counting; (3) using a pre-algebraic description to compute the total number of combinations. We consider that the justification of these initial techniques is based on the necessity to describe the sample space of all possible combinations before counting the total.

In this first phase, it is important that, once students have predicted the total number of codes, they may explain the techniques and models used and justify their use and the resulting answer. It will not be the teacher the only one responsible for validating their answers, as students can check manually by using the padlocks to simulate all the possible combinations. By comparing the different models used by the students, especially the more informal ones, the resulting question is to explore other techniques to compute the total without writing the whole list one by one. At this stage, we expect that students may identify and debate the critical variables in the system to model. In particular: *How to name the cells in the padlock and the other elements (numbers, letters, symbols, etc.)? How many symbols can we have in a cell? Can the elements be (or not) repeated? Is it important the order in which we enter each element?* These are some of the derived questions possible to be posed in this first phase.

In a second phase, teachers brought in four new padlocks with some variations from the initial five. The devices were the same, but they introduced new conditions on the composition of the password:

Padlock 6: It is padlock number 1, but we know that the correct password does not have any repeated number. The number of passwords is then $10 \cdot 9 \cdot 8 \cdot 7$.

Padlock 7: It is padlock number 2, but we do not know how many buttons should be pushed. The correct password can have between 0 to 10 active buttons. The number of passwords is $\binom{10}{0} + \binom{10}{1} + \cdots + \binom{10}{10} = 2^{10}$.

Padlock 8: It is padlock number 3, but we know that the day and the year of the correct password does not have any repeated number.

Padlock 9: It is padlock number 5, but we know that the correct password does not have any repeated direction.

The main question to be presented to the students is Q_2: *Can we use the same kind of counting techniques to find the total number of security codes for any of these new padlocks?* The main purpose of this stage is to test the *validity* of the modelling techniques and the resulting models considered in the previous situation and to understand the changes the new padlocks introduce in the system. Moreover, it is a way to discuss the *scope* of the models by offering students an extended system. In some sense, we could say that this second phase aims to strengthen the modelling praxeology. When trying to apply the initial techniques to the new padlocks, we expect students will make the *logos* part more explicit (what they did, what does and does not work now, why, etc.).

Once the students have all the computations for the nine different padlocks (five with new restrictions and four new ones), we expect the next following questions to appear Q_3: *Are there any formulas that could simplify the total counting of combinations? Are these formulas specific of the "kind of padlock" we want to understand?* We then expect some online search for possible formulas and finding difficulties to interpret them. In this stage, teachers will give some explanations to institutionalise some of the knowledge about combinatorics, to unify the terminology, and to help them to look at the similarities among the different techniques used and results found.

The third phase arrives with an important extension of the initial situation, moving beyond the padlock's problematic presented initially. In this sense, all the work built in the previous phases is now part of the system to model. The question to address with the students is Q_4: *Can we use the same formulas and counting techniques to solve different problems beyond those involving (only) padlocks?* This phase will start when students search for information "outside", that is, possible formulas that exist for the total number of possible combinations. They will probably find information about other contexts that require combinatorial knowledge. At this phase, teachers plan to present a list of other contexts (maybe some of them proposed by the students), with examples of situations including different combinations of elements to count. At this stage, we expect students to analyse the new systems proposed and make decisions on the value of the variables that characterise the combinatorial situations, and on the model to be used. This work requires associating the modelling *praxeologies* developed in the previous stages with the new situations or systems to model.

We are now presenting the results obtained in implementing the SRP with two groups of Grade 10 students during the academic year 2020-2021.

Results from the analysis of the experienced SRP about padlocks

According to the four phases of the didactic engineering process, the *in vivo* analysis was carried out during the SRP implementation through discussions between the two teachers and the research team. During this phase, the team of teachers and researchers decided on the steps to follow depending on the students' productions and classroom dynamics. We are not including this phase here but focusing on the last phase about the *a posteriori* analysis.

The collection of data for the *a posteriori* analysis includes all the students' productions: diaries, reports, recorded oral presentations, test and answers to a survey about the different aspects of the SRP and the students' description of their experience. It also includes all materials produced by the teachers and a shared logbook they filled in at the end of each session, where they record the activities carried out, parts of the joint class discussions and some specific episodes they noticed. These data constitute the evidence used to support our analyses.

First phase: SRP generating question and first modelling process

In the first sessions, teachers gave each working team a different padlock. There were five padlocks (see Figure 1) to analyse, one per team. Teachers asked students to address questions about which padlock is safer (Q_0). Although the first reaction of most teams was trying to find the correct passwords, in the first shared discussion, the students concluded that the lock is more secure when it is more difficult to enter all the passwords. Therefore, they all agreed on trying to count how many passwords could be entered in each lock. Teachers asked each team to answer Q_1 for the padlock assigned: *How many combinations does your padlock have?*

All teams started by computing, manually or using *Excel*, all the possible combinations. Then, they looked for a technique to make this computation faster by thinking about how to count them without explicitly writing all the combinations. For example, *Team A* (Figure 2), which worked with lock number 2 (Figure 1, padlock #2), wrote all the combinations in a spreadsheet. They started by typing 10 columns, which would represent the first digit of the combination. Using the "drag a cell" tool, they automatically typed all the numbers included in each column. They manually removed the combinations numbers that had a repeated digit. They noticed that some combinations were equivalent because they corresponded to the same password: 012, 102, 201, 021, 120 and 210 (in red). They started to paint all the equivalent combinations in the same colour. Finally, they did not paint all the combinations, since they saw that, in all cases, they always painted 6 numbers.

0	1	2	3	4	5	6	7	8	9
"012"	102	201	301	401	501	601	701	801	901
"013"	103	203	302	402	502	602	702	802	902
"014"	104	204	304	403	503	603	703	803	903
"015"	105	205	305	405	504	604	704	804	904
"016"	106	206	306	406	506	605	705	805	905
"017"	107	207	307	407	507	607	706	806	906
"018"	108	208	308	408	508	608	708	807	907
"019"	109	209	309	409	509	609	709	809	908
"021"	120	210	310	410	510	610	710	810	910
"023"	123	213	312	412	512	612	712	812	912
"024"	124	214	314	413	513	613	713	813	913
"025"	125	215	315	415	514	614	714	814	914
"026"	126	216	316	416	516	615	715	815	915
"027"	127	217	317	417	517	617	716	816	916
"028"	128	218	318	418	518	618	718	817	917

Figure 2. Partial view of the list with all the combinations proposed by *Team A*

Once they had an answer for Q_1, they prepared their report to explain the process followed: mainly the questions addressed, the solutions found, and how they had reached these answers using an arithmetical operation. Once the reports were ready, the teacher started a whole group discussion on the work done, avoiding validating their answers. All the teams used the complete list of combinations as a validation means. In the whole group discussion, there were many questions in common among the groups. For instance: *How many cells and how many elements have each cell? Can the elements be repeated? Is it important in which order to put each element?*

All the working teams concluded with a correct answer for their padlock, since they had the complete list of combinations to check the proposed arithmetical operation, although some were more difficult than others. One team – *Team D* –, which had the *directional* padlock (Figure 1, padlock #5) explained to the class that they had looked up for existing answers on the web. They shared with the class that they had found a formula n^m that worked for them, but they could not explain exactly why.

Moreover, one padlock was different from the rest, as it worked by pushing the buttons (Figure 1, padlock #2). This padlock was assigned to *Team B*, made of the students with the best marks in the subject. This team was unable to compute all the combinations in the spreadsheet (as it also happened with *Team A* in Figure 2). When they saw that this initial technique was not useful (or reliable) enough, they changed the technique to model the padlocks' combinations. Figure 3 shows the initial arithmetical model they produced (similarly to other groups) and the development of this model needed to compute the total number of codes for padlock #2. They wrote in the report:

> In summary, we know that in the first cell there can be 10 numbers, in the second cell there can be 9 numbers, and in the third cell there can be 8 numbers. If we multiply these three numbers, we will have 720 possible combinations.

> But the combinations are 720 if the order matters, for example with 1/2/3 there are 6 possible combinations if the order matters, but in the case that it doesn't matter there is only 1. This is what happens with our padlock, as the order is not relevant, for every 6 combinations there is only 1. So, we divide 720 into 6 and get 120 possible combinations.

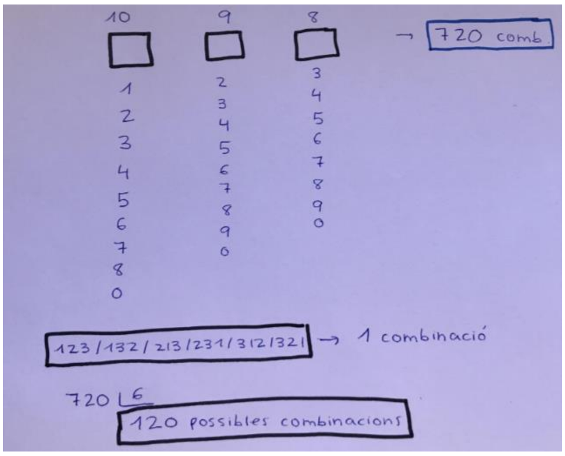

Figure 3. New technique proposed by *Team B* for computing the total of combinations

In the class discussion during this previous stage, the teachers raised the need to set up a common way to refer to several elements that emerged during the modelling process. Students initially used various expressions to communicate their hypotheses, express the variables characterising the system to model (cell, code, combination, order, repetition, etc.), and explain the techniques used and the resulting models to compute the total amount of combinations. It was necessary to establish a common and specific terminology to refer to the mathematical activity they were developing. Using the ATD terminology, we can say that teachers had to introduce new elements of the modelling praxeology to agree on a common *logos*. Thus, they introduced "combination" for the padlock security code, "cell" for each element of a concrete combination, "elements" of a cell for the group of values that can be entered or selected in a cell. Although this terminology was not immediately assimilated and well used by the students, it made it possible to improve the common discussions on the working teams' responses and justifications. To facilitate its adoption, the teachers constantly referred to this agreed vocabulary. They also used it in the institutionalisation of the partial results proposed by the students and the new questions that arose. Here is an example of the teachers' proposal:

> Padlocks admit *combinations*. *Combination* will be the word used to indicate a possible password. We only use *password* for the correct combination. All combinations are made up of *cells*, which are the physical spots we can use to select *cell-elements*. For example, in padlock number 1, we have 4 *cells* and 10 elements in each cell.

Second phase: extending the models' scope

When all the teams finished the first modelling process, the teachers presented the new question Q_2: *Can we use the same kind of counting techniques to find the total number of codes for any of these padlocks?* This time, they proposed students to work with the same five padlocks with constraints and four new padlocks that were also distributed among the teams.

As they used the previous models, students were quicker in analysing the padlocks, determining the values of the variables, and deciding if the padlock was (or not) a similar case to one previously modelled. As mentioned before, this second stage helped students extend and discuss the *scope* of the mathematical models. We can now say that this second phase allowed to strengthen the students' modelling *praxeologies* by making their justification more visible. Students at that time could describe the techniques used, explain why they worked and developed them to get more efficient techniques.

Once the students had all the computations of the total number of combinations for the nine different padlocks, the teachers raised questions about *the existence of any formula that could simplify the total counting of combinations* (Q_3). Most of the students proposed a classification of the padlocks according to the models they used to compute their total number of combinations. Figure 4 shows the classification proposed by *Team A* that classified the padlocks into three groups they named as: "Raised padlocks", "Dividing padlocks" and "Factorial padlocks". In particular, *Team A* then explained the arithmetic calculation that allowed them to compute the combinations and suggested a general formula for each group of padlocks. The students directly established some formulas, but some groups also looked up these formulas on the internet.

Classification of all padlocks according to the resolution method

Name of the group of padlocks	Padlocks in the group	Calculation of the number of combinations of each padlock	Proposed formula
The raised padlocks	1	10 · 10 · 10 · 10 = 10.000 combinations	n^m n = number of cell elements m = number of cells *unless the number of cell elements is different, we use the multiplication of the number of cell elements (as in padlock 3)
	3*	4 · 10 · 12 · 10 · 10 = 48.000 combinations	
	4	10 · 10 · 10 · 10 · 10 = 100.000 combinations	
	5	4 · 4 · 4 · 4 = 256 combinations	
The dividing padlocks	2	10 · 9 · 8 = 720 / 6 = 120 combinations	**n!** is the number of cell elements multiplied by the next number in descending order. **m!** is the number of cells multiplied by the next number in descending order. $\frac{n!}{m!(n-m)!}$
	7	10 · 9 /1·2 + 10 · 9 · 8/1 · 2 · 3... = 1.013 combinations	
The factorial padlocks	6	10 · 9 · 8 · 7 = 5.040 combinations	n · (n - 1) · (n - 2) · (n - 3) ... n = number of cell elements *unless the number of cell elements does not change (as in padlock 8)
	8*	4 · 9 · 12 · 10 · 9 = 38.880 combinations	
	9	4 · 3 · 2 · 1 = 24 combinations	

Figure 4. Classification of the padlocks proposed by *Team A*, translated from the original work of the students

Third phase: the "recursiveness" of the modelling process

After the first two phases of the SRP, a masterclass was planned. Each of the teachers was in charge of presenting the master class to each group. They started reviewing the students' padlocks classification and recollecting the explanations given to find the number of combinations. It was the first occasion for the teachers to validate the solutions officially. Once this was done, they asked the question: *How can we elaborate a general technique to find the number of combinations for each padlock?* They also posed the following derived questions, that have different answers depending on the padlock chosen: *Do two combinations with the same elements but ordered in different ways count as two combinations or as only one? Can two or more elements be repeated in each combination? How many cells does each combination have? How many elements can we put in each cell?* The teachers thus institutionalised the terminology of combinatorics, techniques, and general formulas to quickly calculate the number of combinations (Figure 5).

Let m mean the number of cells and n the number of elements in each cell:		
 All combinations allowed	• The order matters • Elements can be repeated • *Variation with repetition*: $VR_{n,m} = n^m$	$VR_{10,4} = 10^4$
 Only combinations without repeated elements	• The order matters • Elements cannot be repeated • *Variation without repetition*: $V_{n,m} = \frac{n!}{(n-m)!}$	$V_{10,4} = \frac{10!}{(10-4)!} = \frac{10!}{6!}$ $= 10 \cdot 9 \cdot 8 \cdot 7$
 Only combinations without repeated elements and $n = m$	• Same case than before but with $n = m$ • *Permutation*: $P_n = n!$	$P_4 = 4!$
 Combinations pressing 3 of 10 numerical keys	• The order does not matter • Elements cannot be repeated • *Combination*: $C_{n,m} = \binom{n}{m} = \frac{n!}{m!\,(n-m)!}$	$C_{10,3} = \frac{10!}{3!\,(10-3)!}$ $= \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{2 \cdot 3}$

Figure 5. Institutionalisation of the different cases and formulas in the masterclass

At the end of the session, the teachers proposed the students to review their previous answers and associate them with the now institutionalised classification and methods. They also provided a list of typical combinatorial problems in different contexts. At the beginning of each combinatorics problem, they suggested students to relate its characteristics and resolution to the types of padlocks studied. Then, they asked the students if they had found some other situations in which they could apply these new formulas, raising the last question, Q_4.

After the master class, the students answered an individual multiple-choice test. This test contained ten questions with different combinatorics situations (ice cream flavours, coloured T-shirts, etc.) but always with the same structure: "How many different ways can we count...". The next day, the teachers gave the grades and a week later, the students who wanted were allowed to take a second attempt at a similar test.

Discussion and conclusions

A modelling process in combinatorics

Combinatorics problems mainly address questions concerning the counting of complex collections of items. Modelling these collections is a critical step in all the procedures needed to solve them. However, the modelling process usually requires more than one model of the initial situation. We can talk about *recursive modelling* in the sense that a second-step modelling process starts by considering the first model as a new system to model. The inquiry process about padlocks' security presented here illustrates this situation, summarised in Figure 6. The first extra-mathematical system is composed of different types of padlocks and the question raised is about the number of combinations needed to open them. Students spontaneously consider a first model of these combinations in the form of a list of possibilities using an *Excel* table or directly writing all the possibilities (Figure 2). This first model reveals many limitations, especially to validate the final result (*How to know if one combination is missing?*). Teachers intervene to help elaborate a second type of model of the initial system by introducing some terminology to describe the collections of items to count (cells, elements, combinations) and determine their characteristics (order, repetitions, etc.). Students can then propose different numerical formulas with numbers instead of letters (Figures 3 and 4) that play the role of what can be called *intermediate models*. Teachers then institutionalise general algebraic formulas (some previously found by the students) as a third type of model to unify the work done by the teams with the different padlocks (Figure 5). However, the modelling process does not stop here.

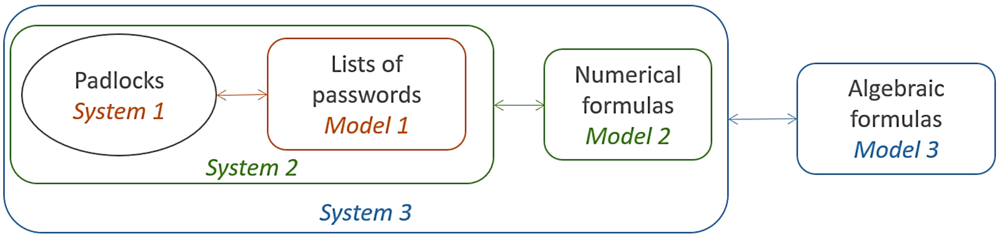

Figure 6. Recursiveness of the modelling process

It is worth noticing that the intermediate models are critical for explaining and validating both the model proposed and the results they help to obtain. These intermediate models thus acquiring a *technological role* in the final combinatorics *praxeology* centred on the algebraic formulas. What did not appear in the experimented SRP was specific work with the algebraic formulas, which would lead to extend the praxeology to include more complex systems or collections to count – a task that exceeded the curriculum exigence of the course. However, the fact that the initial system is made of material objects that can also be handled played a crucial validating role. At the end of the process, students could time how long it took to open each padlock and present it in a short video as a final form of validation.

A second important aspect to mention is that, once students have used the different models to determine the number of possible combinations for each padlock, *they started using padlocks as models* for a variety of new counting problem situation proposed by the teachers. In other words, in a new counting situation (a system including a collection of items to count), students started using the padlocks as concrete references for the new counting problem. This use of padlocks allowed them to characterise the collection to then associate a formula to count its elements. In this second type of situations, the padlocks appear as intermediate models between the system (collection of items to count) and the algebraic formula (see Figure 7).

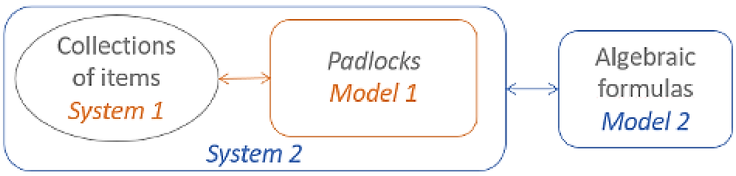

Figure 7. Padlocks acting as intermediate models

Traditionally, the intermediate models used to solve combinatorial problems are the expressions that characterise the collections: variation, permutation, combination. In the students' inquiry process, the padlocks play this intermediate role and help "materialise" each type of situation. The padlock as a model is richer in terms of properties than a simple verbal expression. The case is similar to the one described in Serrano et al. (2010): there is a process of progressive mathematisation of the successive "systems". However, in the end, padlocks appear as representatives of the types of situations considered in combinatorics problems. We thus have an example of a modelling process where the mathematisation is not linear, and extra-mathematical systems – padlocks – end up acting as models of mathematical situations. The interactions between extra-mathematical and intra-mathematical modelling are more complex than the ones described by Serrano et al. (2010).

Promoting a modelling process through a SRP

Our research questions deal with the relationship between combinatorics and modelling practices and the analysis of proposals to promote the construction of combinatorics models in secondary school. For the first question, the SRP about padlocks illustrates how modelling *praxeologies* to approach combinatorics problems rely on the construction of a sequence of models, leading to a progressive mathematisation process. It also shows another phenomenon that is not specific to combinatorics but plays an essential role in this case: the use of "typical situations" (here represented by the padlocks) to initiate the modelling process with new types of systems, in other terms, the use of extra-mathematical systems to model mathematical systems.

The second research question is about the design, implementation, and analysis of a teaching proposal to promote modelling processes in combinatorics. Since the works of García et al. (2006) and Barquero (2009), we know that SRPs tend to generate inquiry processes where modelling *praxeologies* play a crucial role. The SRP about padlocks is not an exception. The openness of the generating question and the students' freedom to explore the initial systems appear to facilitate the emergence of spontaneous modelling processes that teachers contribute to developing. In the second implementation of the SRP that we consider in this paper, teachers were less directive than in the first one. In particular, they let students carry out the first exploration of the combinations by counting them using paper and pencil or *Excel*. This strategy facilitated the emergence of the first models and enriched the initial system. As a second improvement, the teachers enabled the students to search for answers and information about padlocks outside the class. This helped them find some of the formulas finally proposed to the class. We thus see that the elaboration of models in a modelling process can also be nourished by considering external information that does not only arrive from the teacher.

SRPs are still a new instruction format in secondary education, and the conditions for their dissemination as normalised activities need further research. However, as other ATD investigations have shown (Jessen et al., 2020), they seem appropriate to foster the development of modelling processes as a means for the inquiry they promote. The case of combinatorics is not usually considered from a modelling perspective, maybe because of the difficulty to include the recursive vision of modelling and the complex relationship that extra-mathematical and mathematical systems play in this process. Our research aims to link both problems to improve the implementation of SRPs and the teaching of combinatorics as a modelling activity in secondary education.

Acknowledgements

This research was carried out thanks to "Col·legi Natzaret" and the Spanish ministry projects RTI2018-101153-B-C21 and RTI2018-101153-A-C22 (MCIU/AEI/FEDER, UE).

Notes

[1] From now on, we will use *masterclass* as the didactic organization known as *lecture,* as this is the common word used by this school to refer to the didactic moment when the teacher institutionalises some work.

References

Barquero, B. (2009). *Ecología de la modelización matemática en la enseñanza universitaria de las matemáticas* (Doctoral dissertation). Universitat Autònoma de Barcelona, Barcelona. https://ddd.uab.cat/record/63192

Barquero, B., & Bosch, M. (2015). Didactic Engineering as a research methodology: From fundamental situations to study and research paths. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education* (pp. 249-272). Springer.

Barquero, B., Bosch, M., & Gascón, J. (2019). The unit of analysis in the formulation of research problems: The case of mathematical modelling at the university level. *Research in Mathematics Education, 21*(3), 314-330. https://doi.org/10.1080/14794802.2019.1624602

Barquero, B., Bosch, M., & Wozniak, F. (2019). Modelling praxeologies in teacher education: the cake box. In U.T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the CERME 11* (pp. 1144-1152). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Batanero, C., Navarro-Pelayo, V., & Godino, J. D. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics, 32*, 181-199. https://doi.org/10.1023/A:1002954428327

Blum, W., & Leiß, D. (2007). How do students and teachers deal with mathematical modelling problems? The example Sugarloaf and the DISUM project. In C. Haines, P. L. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling: Education, engineering and economics*. ICTMA 12 (pp. 222–231). Horwood.

Borromeo Ferri, R. (2007). Modeling from a cognitive perspective: Individual modeling routes of pupils. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modeling: Education, engineering and economics* (pp. 260-270). Horwood.

Bosch, M. (2018). Study and Research Paths: A model for inquiry. *Proceedings of the International Congress of Mathematics* (pp. 4001-4022). ICM.

Chevallard, Y. (1989). Le passage de l'arithmétique a l'algébrique dans l'enseignement des mathématiques au collège – Deuxième partie. Perspectives curriculaires: la notion de modélisation [The transition between arithmetic and algebra in the teaching of mathematics at secondary school level – Second part. Curricula approaches: The notion of modelling]. *Petit x, 19*, 45-75.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique [The analysis of teaching practices from the anthropological theory of the didactic]. *Recherches en Didactique des Mathématiques, 19*(2), 221-266.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (ed), *Proceedings of CERME 4* (pp. 21-30). Fundemi IQS.

Chevallard, Y. (2015). Teaching mathematics in tomorrow's society: A case for an oncoming counter paradigm. In S.J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 173-187). Springer. https://doi.org/10.1007/978-3-319-12688-3_13

DeGuire, L. (1991). Permutations and combinations: A problem solving approach for middle school students. In M. J. Kenny & C. R. Hirsh (Eds.), *Discrete Mathematics across the curriculum, K-12: 1991 Yearbook* (pp. 55-58). National Council of Teachers of Mathematics.

Dubois, J. G. (1984). Une systematique des configurations combinatoires simples [A system of simple combinatorial configurations]. *Educational Studies in Mathematics, 15* (1), 37-57. https://doi.org/10.1007/BF00380438

English, L. D. (1993). Children's' strategies in solving two- and three-dimensional combinatorial problems. *Journal for Research in Mathematics Education, 24*(3), 255-273.

English, L. D. (2005). Combinatorics and the development of children's combinatorial reasoning. In G.A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 121-141). Kluwer Academic Publishers.

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *ZDM–The International Journal on Mathematics Education, 38*(2), 143-162. https://doi.org/10.1007/BF02655886

García, F. J., Gascón, J., Higueras, L. R., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *ZDM–The International Journal on Mathematics Education, 38*(3), 226-246. https://doi.org/10.1007/BF02652807

Jessen, B., Otaki, K., Miyakawa, T., Hamanaka, H., Mozoguchi, T., Shinno, M., & Winsløw, C. (2020). The ecology of study and research paths in upper secondary school. In M. Bosch, Y. Chevallard, F. J. García, & J. Monaghan (Eds.), *Working with the Anthropological Theory of the Didactic in mathematics education: A comprehensive Casebook* (pp. 118-138). Routledge. https://doi.org/10.4324/9780429198168

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modeling in mathematics education. *ZDM–The International Journal on Mathematics Education, 38,* 302-310. https://doi.org/10.1007/BF02652813

Kapur, J. N. (1970). Combinatorial analysis and school mathematics. *Educational Studies in Mathematics, 3*, 111-127. https://www.jstor.org/stable/3481871

Lockwood, E. (2013). A model of students' combinatorial thinking. *Journal of Mathematical Behavior*, *32,* 251-265. https://doi.org/10.1016/j.jmathb.2013.02.008

Niss, M., & Blum, W. (2020). *The learning and teaching of mathematical modelling.* Routledge. https://doi.org/10.4324/9781315189314

Perrenet, J., & Zwaneveld, B. (2012). The many faces of the mathematical modeling cycle*. Journal of Mathematical Modelling and Application, 1*(6), 3-21.

Roa, R., Batanero, C., & Godino, J. (2003). Estrategias generales y estrategias aritméticas en la resolución de problemas combinatorios. *Educación Matemática 15(2)*, 5-25.

Serrano, L., Bosch, M., & Gascón, J. (2010). Fitting models to data. The mathematising step in the modelling process. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 2185-2196). Institut National de Recherche Pédagogique and ERME.

Wozniak, F. (2012). Des professeurs des écoles face à un problème de modélisation: Une question d'équipement praxéologique. *Recherches en Didactique des Mathématiques, 32(1),* 7–55.