Mathematical modelling in the teaching and learning of mathematics: Part 2

Modelação matemática no ensino e aprendizagem da matemática: Parte 2

Susana Carreira
Faculdade de Ciências e Tecnologia, Universidade do Algarve
UIDEF, Instituto de Educação, Universidade de Lisboa
Portugal
scarrei@ualg.pt

Werner Blum
Institute of Mathematics, University of Kassel
Germany
blum@mathematik.uni-kassel.de

Hardworking ants and the power of mathematical models

The second part of the thematic issue of the Quadrante journal dedicated to *Mathematical modelling in the teaching and learning of mathematics* is completed, fulfilling an editorial decision to dedicate the entire 30th volume to this theme, which stems from the high level of participation from contributors throughout the world. The strong involvement of many authors resulted in a great diversity of research questions, theoretical approaches (from didactical to cognitive, sociocultural, or epistemologically driven perspectives), educational levels and contexts, mathematical contents, and research methodologies. Interesting results pertinent to the field of mathematical modelling in education are a product of this.

The current issue brings timely and refreshing thinking about the powerful role of mathematical models in society, in the various scientific disciplines, and in the everyday world (including our current pandemic-torn world). It is never too much to recall the words of Pollak (2007, p. 119): "if you are going to model phenomena from outside of mathematics, you . . . have to understand those phenomena. This requires people who teach mathematics to understand things that they normally are not required to understand".

We all know the ant as a symbol of hardwork, and it may well describe one of the aspects of this thematic issue. Beyond the metaphorical hint, it should be noted that the ants'

https://doi.org/10.48489/quadrante.26132

organization in foraging is a phenomenon that can be modelled mathematically. A paper entitled "A modelling framework for understanding social insect foraging" (Sumpter & Pratt, 2003) clearly elucidates the centrality of representations and mathematical thinking, whose use is not always obvious in school mathematics. For example, the mathematical modelling in this biological context involves finding the zeros of the function (in which b and c are parameters): $f(x) = bx - cx^2 - \frac{x}{1+x}$. The solutions lead to the conclusion that in colonies with more than 1000 ants the discovery of a food source by a single ant will, in the long term, result in many ants foraging on the food source. By contrast, for smaller colonies, the same small event will not result in an increase in foraging.

Like this sketchy case illustrative of the countless uses of mathematical modelling, the collection of articles that compose the second part of the thematic issue contain examples of the use of *descriptive*, *predictive*, and *prescriptive* (or normative) models in quality mathematics teaching and learning (Blum, 2015; Niss, 2015; Niss & Blum, 2020). Several instances of boundary crossing pedagogies (Leung, 2020), as well as bridging objects across disciplines (Doig et al., 2019), are reported and analysed – some on the topical problem of the spread of infectious diseases – together with studies that focus on the modelling process, and how it may be promoted in schools, universities, teacher education programs, and classroom practices.

An overview of the second part of this thematic issue

This second part includes 14 research articles, which can be grouped around three overarching topics: i) the role of models and model analysis in scientific domains and social areas; ii) mathematical modelling as a vehicle for the learning of mathematics, at different educational levels; iii) learners' modelling activities, including future teachers, and the teaching practice of mathematical modelling.

It opens with the inspiring article by *Blomhøj* and *Niss*, addressing one aspect of mathematical modelling competency that has gained less attention in research: the analysis of extant mathematical models, originating from other scientific disciplines. In their theoretical work, the authors examine the various processes and actions that are required to carry out such an exploratory analysis and show how it is closely related to the constructive modelling process. Based on a condensed exposition of the essential elements of an economics model and a chemistry model, the authors examine the "archaeology" of both models and conclude that analysing extant models is no less demanding than constructing new models. A significant result from their essay is the proposal of a novel Diagram for Model Analysis. Furthermore, and based on the Danish KOM framework, the article anticipates didactical consequences of model analysis for promoting the analytic side of modelling competency in mathematics teaching and learning.

Also framed within the reality of the Danish mathematics curriculum, *Jessen* and *Kjeldsen* analyse the didactic transposition of scientific and scholarly mathematical modelling practices to understand how those are framed within Danish upper secondary mathematics. The authors develop their argument by contrasting examples of mathematical models produced in economics and biology with the way in which official curricular guidelines, textbooks and exam questions deal with mathematical modelling. They identify similarities and differences between scholarly knowledge and the school knowledge enacted in the mathematics curriculum. For example, disparities come to the fore when looking at text-books where the examples and exercises display real situations that are already mathematized, with the main emphasis being placed on the mathematics that students should apply and put into practice. Adding to previous studies, this article draws attention to the fact that modelling practices and scholarly knowledge undergo considerable changes in the didactic transposition, ending up by restraining the learning of desired modelling practices and making interdisciplinarity highly constrained.

The article focusing on *spin-off* learning, by *Andresen*, reports a case study of students modelling epidemics with differential equation systems in a master's course on mathematics education. The idea of *spin-off* learning refers to how the participants, during the modelling process, acquired knowledge about the epidemiological situation they were modelling. The directions of the modelling activity, defined as expressive and explorative modelling, combined with the Realistic Mathematics Education's theorization of horizontal and vertical mathematization, supported the analysis of the conditions for *spin-off* learning in students' mathematical modelling of epidemics. The data were obtained from the descriptions of the mathematical modelling activities of four groups of students and from the reflections they presented in their written reports. The four projects carried out by the groups concerned epidemiological situations related to the spread of viruses such as Influenza, Ebola, or HIV. One relevant result of the study is that many instances of *spin-off* learning were observed, and this occurred under both expressive and exploratory directions of the modelling process. It was also concluded that modelling projects, if carefully chosen to include the investigation of important societal areas, show great potential for *spin-off* learning also to be expected in school mathematics.

Still relating to the mathematical modelling of epidemics, *Blomhøj* and *Elicer* discuss the exemplary character of epidemiology from three perspectives: subjective, instrumental, and critical. Claiming that one of the strong justifications for integrating mathematical modelling in the school context is the critical justification of mathematical models used in society, the authors propose to examine how the spread of epidemics can serve as an appropriate issue for the teaching and learning of mathematical modelling. In particular, the problem of epidemics exemplifies a domain in which mathematical models play essential roles in political decision-making concerning social risk phenomena. Drawing on the experience obtained from extra-curricular workshops offered to talented high school students, the researchers analyse how students engaged in modelling the dynamics of an epidemic caused by an infectious virus in a closed model population. The observed results show that the challenge of modelling an epidemic has fulfilled the various intended perspectives of modelling. Moreover, the workshops reveal the potential to engage students in reflections on the role and function of mathematical models in social health issues related to infectious diseases, from a socio-critical point of view.

The theoretical work presented in the article by *Højgaard* focuses on conceptually distinguishing between modelling competency and problem-solving competency as these terms are employed in mathematics education. Based on a set of tasks that are carefully described and categorized, the author highlights the core differences, both at an abstract and more concrete level, between the two competencies. The systematic analysis of a range of tasks is followed by Højgaard's reflection on the difficulties and impediments that might explain the fact that it is rare to see some of those types being used in mathematics education. The author concludes that too often such tasks are replaced by oversimplified invitations to mathematization or by pseudo extra-mathematical situations. One of the crucial barriers pinpointed is the impact of external written examinations; the author's closing comment suggests that changing the written examinations would contribute to taking mathematical modelling competency seriously in mathematics education.

Moving to the second overarching theme, mathematical modelling as a vehicle for learning, *Almeida*, *Silva*, and *Borssoi* report on an empirical study in which the broad concept of experimentation, including the use of technological tools, is placed at the heart of the mathematical modelling activity. Their study aimed to find out how experimentation mediates the students' learning within mathematical modelling activities. It involved higher education students from different undergraduate courses, one in mathematics and two in engineering. One of the problems given to the mathematics undergraduates was to model the temperature variation inside a car that, after being in the sun, was parked in the shade with the windows open. The other problem proposed to the engineering undergraduates was about modelling speed radars and the students investigated how fast a car should travel to beat a speed camera. The results, from the two groups of students, provided evidence that the technology-based experimentation included three essential components: structuring the problem, planning and running a computational experiment, and discussing the results. Despite the differences between the two groups in the way they used experimentation with technology, it was decisive in creating conditions for students' learning.

The study presented by *Moreno-Sandoval* and *Alvarado-Monroy* concerns the use of participatory digital simulations in modelling the effects of access to medical care in Mexican indigenous peoples. The authors adopt the theoretical perspective of Models and

Modelling, according to which the modelling process is the generator of learning. Based on participatory simulations conducted through the programmable modelling environment NetLogo, a modelling development sequence was designed to promote the covariational reasoning of secondary school students. Under a design-based research methodology, five tasks were implemented and analysed. To a large extent, the results observed in the evolution of the students' developmental levels of covariational reasoning can be attributed to the characteristics and principles of the modelling sequence. Starting from the reality that they were able to experience through NetLogo, the students interpreted, expressed, and quantified covariation as linked to their interpretation of the context situation.

Also following the Models and Modelling perspective, *Aymerich* and *Albarracín* adopted the design principles of modelling eliciting activities in a study aimed at fostering secondary students' learning of statistical knowledge, namely the development of the concept of variability. The activity dealt with the variability of the car circulation tax across Spain, and the general purpose was that students should understand the need to measure the variability existing in a large amount of quantitative data. As a result of the study, the authors infer that the students were able to generate their own mathematical models to solve the given situation, by using and combining mathematical concepts they already knew from previous experience. Thus, they conclude that the students successfully started to model large amounts of contextualized data. Moreover, in their model generation, fundamental aspects of the concept of data variability were identified.

Finally, in this group of articles, a research study by *Vásquez*, *Barquero* and *Bosch* is reported. Their work is grounded in the Anthropological Theory of the Didactic and its conception of mathematical modelling. The article presents the design and implementation of a Study and Research Path for the learning of combinatorics in compulsory secondary education. The students were given several different padlocks and were asked to find out which type would be most secure. The authors analysed the students' construction of combinatorial models to represent the ideas produced through the interaction with the padlocks. After the simulation and validation of the elementary combinatorial models created by the students, the models were generalized to explore other combinatorial systems beyond padlocks. An interesting result of the study is that once students use models to determine the number of possible combinations for each padlock, they start using the different padlocks as concrete models that could be applied to a variety of new counting situations. As the authors note, despite being a still new instruction design in secondary education, the Study and Research Path proved appropriate to foster modelling processes, particularly in the case of combinatorics where a complex relationship between extra-mathematical and mathematical systems is at play.

A third set of articles referring to empirical studies on the modelling activity of students, as well as future teachers, begins with the contribution of *Schneider*, *Borromeo Ferri* and

Ruzika. They investigated whether and to what extent having knowledge about the ideal-typical modelling process influences the structure of the modelling routes of individuals. For this purpose, they collected data from two groups of 10th grade students solving a modelling problem, one having received a specific instruction on the typical modelling process and the other without such an instruction. Using a research instrument called the Modelling-Activity-Interaction-Tool (MAI-Tool), the researchers conducted a mixed-method study in which a set of hypotheses was established. Taking advantage of the MAI-Tool, it was possible to examine the occurrence and duration of the phases in each individual modelling route and therefore to compare individual cases from each group. Through statistical tests, the research hypotheses were then tested, by comparing the two groups. The results show that they differed significantly. The individuals with instruction about the ideal-typical modelling process took more time to understand the task, whereas the students without such instruction created a model after a short time. The first group of students also engaged more often and for a longer time in the mathematical world, in formulating a mathematical model and working on it.

Ludwig and *Jablonski* examine students' modelling processes in the context of the MathCityMap project. The authors focus on the steps of the modelling process which takes place in math trails consisting of certain real-world situations accessible by the students outside their classroom and associated modelling problems. A typical example is determining the size or weight of real objects such as a stone monument in a park. In a qualitative pilot study involving two pairs of university students in a mathematics education course, the study compared the solution processes of those who worked with real objects outdoors, using the MathCityMap app, and of those who worked with photos of the real objects inside the classroom. The authors expected that the processes carried out by the two groups would prove to be different. The results show that, in both cases, the students proceeded successfully through the structuring/simplifying and the mathematizing steps. One important difference was that the results were only validated in the outdoor setting. The authors make inferences about the potential of the MathCityMap app for students to gain competency on performing the steps of the modelling cycle.

A case study with preservice teachers on the meanings of modelling is presented in the article by *Esteley* and *Cruz*. The aim was to understand the meanings assigned to mathematical modelling and to see how those meanings change because of the preservice teachers' work on modelling tasks. The study was developed within an action research project linked to the implementation of mathematical modelling in a university course on space geometry. When analysing the work of the future teachers during the different activities, a change in the meanings related to mathematical modelling was observed. The meanings produced evolved from a mere association of modelling with examples or applications of given models to the awareness of the sub-processes involved in mathematical modelling. Not only were the participants able to identify such sub-processes but they also offered examples to illustrate them. The conclusions highlight the fact that changes in meaning are not independent of the activities and interactions that are promoted in the classroom, in the sense that meanings are produced and modified in their interplay with the activities and the person(s) who intervene, namely the teacher.

Alwast and *Vorhölter* present an empirical study undertaken with future teachers on the development of their noticing competencies while teaching mathematical modelling. The authors conceptualise noticing as the use of a particular lens to view situations, which is specific to each profession; therefore, noticing in a modelling context will require a modelling-specific lens. In their article, they report on a case study with three future teachers on the development of their noticing competencies after the application of a video-based instrument composed of two staged videos. The videos simulated real classroom situations, displaying a group of four students in ninth grade working on a modelling problem. For the data content analysis, growth indicators of noticing competencies were used. The indicators covered the following aspects: reasoning about students' thinking, linking interpretations of students' work to specific details of the modelling situation, and considering the influence of individual students on the group. The results reveal that the three participants developed differently across the growth indicators. In general, the growth indicators proved to be useful to understand more precisely in which aspects the development of noticing competencies took place.

This cluster of articles closes with the qualitative research study from *Araújo* and *Lima* concerning the nature and aim of the examples used by an experienced teacher during his interventions to support students' progress in modelling projects. Their initial conjecture was that the teacher's extensive theoretical and practical experience with modelling projects will contribute to the use of examples that are different in nature and aim, when compared to those commonly used in mathematics classes. Drawing on the concept of exemplarity and relating it to distinctive types of modelling tasks in different modelling perspectives, the authors sought to understand the use of examples in the teacher's practice on modelling projects. In the analysis of three teaching episodes, the examples the teacher used to guide the groups in their respective modelling projects were categorized, in terms of aim and nature. It was possible to identify three main purposes for the examples used: didactic, socio-political, and fictional. It was also observed that some examples featured one or more of these simultaneously, depending on the intentions of the teacher as well as his specific real-world knowledge. This reinforces the importance of considering the teacher's previous experience when studying the teaching practices of mathematical modelling.

As a final comment on this second part of the thematic issue, we emphasise the wide scope of the articles that it comprises. They range from theory-driven approaches to the character and role of mathematical models in scientific disciplines and societal fields, to empirical studies on modelling for the learning of mathematical topics and concepts, and on modelling processes undertaken by students and teachers. Along with the first part of this thematic issue, the full collection of articles offers inspiration, pertinent theory and scientific evidence gathered from around the world to claim the essential place of mathematical modelling in today's mathematics education throughout all educational levels.

Acknowledgements

We would like to express our gratitude to all the authors and reviewers for their valuable contribution to achieve the desired quality of this vast thematic issue of the Quadrante journal. Special thanks to Peter Galbraith for his kindness in timely proofreading our editorial. Again, we thank the editorial board of Quadrante for their full cooperation and support. For the guest editors, this has been a stimulating, challenging, and rewarding endeavour.

References

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do?. In S. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 73-96). Springer. https://doi.org/10.1007/978-3-319-12688-3_9

Doig, B., Williams, J., Swanson, D., Borromeo Ferri, R., & Drake, P. (Eds.). (2019). *Interdisciplinary mathematics education: A state of the art. ICME-13 Topical Surveys*. Springer. https://doi.org/10.1007/978-3-319-42267-1_1

Leung, A. (2020). Boundary crossing pedagogy in STEM education. *International Journal of STEM Education*, *7*, 15. https://doi.org/10.1186/s40594-020-00212-9

Niss, M. (2015). Prescriptive modelling – Challenges and opportunities. In G. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education: Research and practice. (International perspectives on the teaching and learning of mathematical modelling)* (pp. 67-79). Springer. https://doi.org/10.1007/978-3-319-18272-8_5

Niss, M., & Blum, W. (2020). *The learning and teaching of mathematical modelling*. Routledge. https://doi.org/10.4324/9781315189314

Pollak, H. (2007). Mathematical modelling – A conversation with Henry Pollak. In W. Blum, P. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 109-120). Springer. https://doi.org/10.1007/978-0-387-29822-1_9

Sumpter, D. J. T., & Pratt, S. C. (2003). A modelling framework for understanding social insect foraging. *Behavioral Ecology and Sociobiology*, *53*(3), 131-144. https://www.jstor.org/stable/4602194