Teacher knowledge and its sources: Perceptions of lower secondary teachers of mathematics

O conhecimento do professor e as respetivas fontes: Perceções de professores de matemática do ensino básico e secundário

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Abstract. Understanding what knowledge domains teachers perceive as the basis for teaching mathematics is important because teachers' perceptions (or beliefs or attitudes) can influence the way they approach teacher development and what they learn from it. The interest of this study is mathematics teachers' perceptions of knowledge needed for teaching mathematics and its sources. Six lower secondary teachers of mathematics (Years 7 to 10) in Tasmania, Australia, completed a questionnaire including eight open-ended items, and the data were content analysed deductively. Mostly a single knowledge dimension was articulated (knowledge of content and mathematics, or knowledge of content) as opposed to the multidimensionality of teacher knowledge. Several sources of teacher knowledge were evident in teachers' articulations including both formal (workshops, conferences) and informal (peer interactions or collaboration) sources, whilst university coursework and educational research sources were absent. Teacher education and development contexts might explain these findings, and implications for them are discussed.
Keywords: mathematics education; perceptions and beliefs; sources of knowledge; teacher knowledge; teacher education and development.

Resumo. É importante compreender quais os domínios de conhecimento que os professores consideram como principais para o ensino da matemática porque as perceções (ou crenças ou atitudes) dos professores podem influenciar a forma como abordam o seu desenvolvimento profissional e o que daí aprendem. O foco deste estudo são as perceções dos professores de matemática sobre o conhecimento necessário para o ensino da matemática e sobre as respetivas fontes. Seis professores de matemática do ensino básico e secundário (7.º ao 10.º ano) da Tasmânia, Austrália, responderam a um questionário composto por oito itens de resposta aberta e os dados foram analisados por meio de análise de conteúdo dedutiva. Essencialmente, foi indicada uma única dimensão do conhecimento (conhecimento de matemática, ou conhecimento de conteúdos), em contraste com a multidimensionalidade do conhecimento profissional do professor. Várias fontes de conhecimento dos professores

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foram evidentes no discurso dos inquiridos, incluindo formais (workshops, conferências) e informais (interações entre pares ou colaboração), sendo que cursos universitários e referências à investigação educacional estiveram ausentes. Os contextos de formação e desenvolvimento profissional dos professores poderão explicar esses resultados e são discutidas as respetivas implicações.
Palavras-chave: educação matemática; perceções e crenças; fontes de conhecimento; conhecimento do professor; formação e desenvolvimento profissional do professor.

Introduction

For decades now, what teachers would know and understand to teach the content effectively, and create positive impact on student learning, has been a focus of interest for researchers and teacher educators. As a key constituent of effective or good teaching (Mosvold & Fauskanger, 2014), the construct of teacher knowledge has permeated the literature. The construct has often been accepted to include different knowledge domains such as content knowledge, curriculum knowledge, and pedagogical content knowledge (e.g., Shulman, 1987; Ball et al., 2008). While there is a considerable body of research on teacher knowledge, what is often missing in this field is how teachers describe the knowledge needed for teaching, and what they perceive as possible sources of that knowledge (with the notable exceptions of Buehl & Fives (2009) and Fives & Buehl (2008) on teachers' epistemological beliefs). The focuses of most existing studies have been on teachers' (both pre- and in-service) beliefs about mathematics or its teaching and learning (Felbrich et al., 2012; Maasepp & Bobis, 2014; Schoen & LaVenia, 2019; Viholainen et al., 2014), and sometimes from cross-cultural perspectives (Bryan et al., 2007; Kaiser & Vollstedt, 2007; Werler & Tahirsylaj, 2022).

Understanding what knowledge domains teachers perceive as the basis for teaching mathematics is important because pre-service teachers' perceptions (or beliefs or attitudes) influence the way they approach teacher education and what they learn from it (Cady & Rearden, 2007; Jong & Hodges, 2015; Richardson, 1996). The perceptions that in-service teachers hold about content and its teaching and learning influence the way they approach professional learning (PL), what they learn from it, and how they change or grow (Mosvold & Fauskanger, 2014; Richardson, 1996). Furthermore, teachers' perceptions about teacher knowledge and its sources can influence their approaches to teaching and learning (Buehl & Fives, 2009; Ernest, 1989; Pajares, 1992; Schoen & LaVenia, 2019). Understanding the content of those perceptions then should be at the centre of teacher education, teacher development (Buehl & Fives, 2009; Cady & Rearden, 2007; Ernest, 1989), and curriculum reforms (Handal, 2003). This can not only inform mathematics teacher educators in enacting a vision of teacher education and development but also can improve the design of teacher education and development programmes in some way. The present study gives a comprehensive insight into perceptions of six lower secondary mathematics teachers of teacher knowledge and its sources (Years 7 to 10, 12-16 years old). The questions that governed the study are:

(1) What are lower secondary mathematics teachers' perceptions of teacher professional knowledge and its sources?

(2) To what extent do lower secondary mathematics teachers recognise teacher knowledge domains and possible sources of them?

In the following sections, I present the conceptual basis for the study, before presenting its methods. Next, findings are presented around research questions, and they are interpreted comprehensively in the discussion section. This is followed by the implications of the findings for the teacher education and development fields.

Conceptual bases for the study, and previous research

Teacher perceptions, or beliefs or attitude

As we describe in Hatisaru et al. (2023), the construct of perceptions encompasses what individuals believe, think, or feel about a relevant construct. In mathematics education and related fields, it is used essentially to mean attitudes (e.g., Hatisaru, 2020a). Synonymous to perceptions, the construct of beliefs "are thought of as psychologically-held understandings, premises or propositions about the world that are felt to be true" (Richardson, 1996, p. 5). When used to describe attitudes, perceptions can be said to be positive or negative whereas when used to reflect beliefs there is no inherent evaluative component (see Hatisaru et al., 2023).

In this study, it is accepted that perceptions are part of teachers' thought structures along with beliefs and attitudes, and they are among the affective outcomes of teacher education, of general educational contexts and experiences (Ernest, 1989). Perceptions refer to a way of understanding or thinking about the knowledge needed for teaching mathematics and how that knowledge is acquired or developed. Whilst I was less interested in whether teachers felt positively or negatively inclined towards the concepts of teacher knowledge and its sources, I was aware that some participants could express their beliefs with attitudinal ingredients by indicating, for example, 'curriculum knowledge is an important part of teacher knowledge'. Therefore, in this study, the concept of perceptions is used to incorporate both beliefs and attitudes.

The construct of teacher knowledge

Scholars in the field of teaching and teacher education have suggested frameworks or taxonomies of the teacher knowledge base to describe the notion of teacher knowledge. In this section, some of the major teacher knowledge models that have been proposed to date are presented; however, elucidation of each of their constituents is not a central focus of the present paper.

In one of his seminal works in this field, Shulman (1987) comprehensively elaborated on the knowledge base for teaching to maintain student learning. According to Shulman (1987), successful teachers connect content and pedagogical skills for different kinds of students, different pedagogical goals, and different levels of difficulty. These dimensions of knowledge are necessary for teachers: content knowledge, general pedagogical knowledge (e.g., strategies of classroom management), curriculum knowledge (e.g., the materials and programs), pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of educational contexts (e.g., the governance and financing of school districts), and knowledge of educational ends, purposes, and values. Echoing framings of Shulman (1987), Grossman (1990) suggested four major dimensions of teacher knowledge as the core of professional knowledge for teaching: general pedagogical knowledge, subject matter knowledge, pedagogical content knowledge, and knowledge of context. Specific to teachers of mathematics, Ernest (1989) considered knowledge as part of teachers' thought structures in the Knowledge, Beliefs and Attitudes of the Mathematics Teacher model. This model identifies not only theoretical knowledge such as knowledge of mathematics and of other subject matter, but several important aspects of practical knowledge for teaching, including knowledge of organisation and management for mathematics teaching; and knowledge of the context of teaching mathematics (both knowledge of school and students taught). In agreement with Ernest's (1989) conceptualisations, according to Fennema and Franke (1992), teacher knowledge includes understanding the underlying process of the concepts, being able to interpret these concepts for teaching, understanding students' thinking, and being able to assess students' learning to make instructional decisions, as well as having general pedagogical knowledge. The model they proposed includes knowledge of content (the concept, procedures, and problem-solving process in maths), knowledge of pedagogy (teaching procedures, planning, management and motivation), knowledge of students' cognition (students' thinking, learning, and difficulties), and beliefs of teachers.

The notion of Pedagogical Content Knowledge (PCK), coined by Shulman (1987), has received significant attention, and it has been used in both framing and describing teacher knowledge and understanding how teacher knowledge influences the effectiveness of the teacher. Grossman (1990) expanded on Shulman's definition of PCK by identifying four components of teacher knowledge: teachers' "overarching conception of the purposes for teaching particular subject matter; knowledge of pupils' understanding and potential misunderstanding of a subject area; knowledge of curriculum and curricular materials; and knowledge of strategies and representations for teaching particular topics" (p. 40). Based on an in-depth study of eight primary teachers, Marks (1990) proposed four components of PCK: "subject matter for instructional purposes, students' understanding of the subject matter (student learning process, typical understandings, common errors, difficulties), media for instruction in the subject matter (i.e., texts, materials), and instructional processes for the subject matter" (p. 4). An et al. (2004) proposed a network of PCK that broadens these mentioned definitions, based on their investigation into PCK of mathematics teachers in middle schools in China and the USA. They defined PCK as the knowledge of effective teaching comprising three components: knowledge of content, knowledge of curriculum (selecting and using appropriate textbooks and materials, understanding the goals of textbooks and curricula), and knowledge of teaching (knowing students' thinking, planning instruction, understanding the modes of presenting instruction). In this network, knowledge of teaching is the main component of PCK; and knowing students' thinking includes addressing students' misconceptions, engaging them in mathematics learning, building on their mathematical ideas, and promoting their mathematical thinking.

Different models of teacher knowledge have been advanced building on the premise of PCK (Shulman, 1987) including Ball et al.'s (2008) Mathematical Knowledge for Teaching (MKT). In brief, MKT distinguishes between subject matter knowledge (SMK) and PCK and refines both. It consists of three domains of SMK: common content knowledge (the mathematical knowledge and skills used in settings other than teaching), specialised content knowledge (the mathematical knowledge and skills unique to teaching), and horizon content knowledge (how mathematical subjects are related in the continuum of mathematics included in the curriculum). The PCK domain includes knowledge of content and students (knowledge of how students think about, know, or learn a particular content), knowledge of content and teaching (knowledge of a mathematical content, idea or procedure and knowing pedagogical principles for teaching that content), and knowledge of content and curriculum.

In summary, what teachers should know and understand in order to teach the content effectively, and create positive impact on student learning, has been a focus of interest for researchers and teacher educators for decades. The various teacher knowledge frameworks mentioned in this section are complementary to a certain degree (Charalambous & Pitta-Pantazi, 2016) and clearly describe the necessary capabilities of teachers to teach the content effectively. They provide the foundation for both developing teacher education programs (e.g., Charalambous, 2015) and guiding teacher learning (e.g., Kazemi & Franke; 2004; Özgün-Koca et al., 2020). Each of these frameworks includes the essential knowledge domains, many of which are empirically well-grounded as they have been identified based on comprehensive empirical work. There is considerable convergence among these teacher knowledge conceptualisations with knowledge of content, PCK, and knowledge of student thinking being common constituents.

I consider the construct of teacher knowledge as multidimensional, consisting of a variety of knowledge domains mentioned above. The teacher knowledge frameworks described here—and possible others—are complementary, whilst they are named differently. To me, knowledge of mathematics and knowledge of mathematics teaching are indispensable for teachers of mathematics, and perhaps they are indistinguishable (Charalambous & Pitta-Pantazi, 2016). In this paper, I content analysed the participating teachers' responses to an open-ended questionnaire to discover their perceptions, informed by the categories of teacher knowledge identified here. I was also open to adding more knowledge dimensions if they should arise within the data.

Prior research on teacher knowledge

Most relevant to this paper, Mosvold and Fauskanger (2013) explored teachers' beliefs about MKT regarding mathematical definitions. Although the teachers believed its importance as an aspect of MKT, they did not find that they need to know mathematical definitions. Mosvold and Fauskanger (2014) investigated teachers' beliefs about horizon content knowledge and found that teachers did not necessarily view this knowledge as an important part of their teaching knowledge. Hatisaru and Collins (2023) investigated the extent to which a sample of pre-service secondary mathematics teachers (PSMTs) recognised the professional knowledge needed for teaching mathematics and their perceptions of its sources. Data were collected by using the same questionnaire used in this paper. The PSMTs mostly mentioned more than one knowledge dimension, placed much emphasis on mathematics content knowledge and mathematics pedagogical knowledge. They expected to gain most of their knowledge through formal preparation within the professional learning system; self-study or interactions with peers were relatively less mentioned as sources of knowledge.

While they did not necessarily investigate perceptions of teachers on teacher knowledge, the notion of what teachers need to know has emerged in the studies examining teachers' beliefs about 'good' or 'effective' teaching. For example, Leong (2015) investigated what beginning teachers think about the characteristics of good teaching and found that having robust content knowledge, classroom management, and motivation were the key attributes of good teaching according to the participating beginning teachers (Leong, 2015). Within a ZDM Special Issue in 2007, based on a cross-cultural study conducting interviews with participating teachers from Australia, Hong Kong SAR, Mainland China, and the USA, Perry (2007) described a group of selected Australian teachers' beliefs about effective mathematics teaching and learning. Bryan et al. (2007) reported similarities and differences in teachers' beliefs from these four nations, and Kaiser and Vollstedt (2007) compared teachers' beliefs from the four nations reported by Bryan et al. (2007) with teachers' beliefs in Europe. The results of these studies showed that there was agreement on some of the qualities of effective mathematics teachers among participating teachers from the four nations. These included competence in mathematics and necessity of in-depth understanding of the curriculum and textbooks, the latter quality being especially mentioned by teachers from Mainland China. All the teachers emphasised that teachers need to know their students and understand students' educational needs. Teachers from the US pointed out teachers' classroom management skills, while this was not expressed by other teachers. Teachers from the US and Australia articulated teachers' ability of listening to students and getting them to express themselves.

Sources of teacher knowledge

Sources of knowledge can be defined as "the domains of scholarship and experience from which teachers may draw their understanding" (Shulman, 1987, p. 5). There have been comprehensive approaches to identifying sources of knowledge for teachers to improve the quality of teaching. A summary of existing models described in this section is presented in Table 1. These models have much in common, while named differently. Shulman's (1987) description of the wisdom of practice recognises the experiences of teachers. Buehl and Fives's (2009) formal preparation and formalised bodies of information reflect Shulman's (1987) educational materials and research on schooling categories, and Richardson's (1996) experience with formal schooling. Ernest's (1989) and Richardson's (1996) learning experiences underline the role of teacher education in knowledge acquisition, both in terms of the content of learning experiences and the mode of content delivery (i.e., teaching style), and they echo Buehl and Fives's (2009) personal experiences theme. Interactions with colleagues is addressed both in Buehl and Fives (2009) and Wilson et al. (2005). In this study, these sources of teaching and teacher knowledge are used as priori coding categories in data analysis to understand perceptions of teachers of the sources of teacher knowledge.

Table 1. Sources of teaching and teacher knowledge

Sources of knowledge for teachers needed for teaching (Shulman, 1987)	**Experiences that influence beliefs and learning to teach and teaching (Richardson, 1996)**	**Ways good teaching is learned (Wilson et al., 2005)**	**Sources of teaching knowledge (Buehl & Fives, 2009)**
Scholarship in subject areas Educational materials and institutional contexts Research on schooling The wisdom of practice	Personal experiences Experiences with schooling and instruction Experiences with formal knowledge	Experience Education Personal reading and reflection Interaction with colleagues	Formal preparation Formal bodies of information Observational learning Collaboration or interactions Enactive experiences Self-reflection

To elaborate on more, Shulman (1987) identified at least four main sources of knowledge for teachers needed for teaching. The first source (scholarship in subject areas) is the content knowledge, understanding and skills that teachers teach. This content knowledge comes primarily from the accumulated studies in the relevant subject area and the historical and philosophical research in the nature of knowledge in that field. A second source (educational materials and institutional contexts) is the materials and structures that are created for teaching and learning. These include curricula; textbooks; tests and testing materials; school organisations; the structure of the teaching profession; and the principles, policies and facts of institutions including professional teacher organisations, state and federal level government agencies. As teachers function within a structure created by these elements, they contain a rich source for the knowledge needed for teaching. The third source (research on schooling) is the existing body of academic literature on understanding the process of teaching and learning. This body of knowledge includes empirical research findings in the fields of teaching, learning, human development, and the philosophical and ethical foundations of education. The research in these areas can be both generic and content specific. The former can be, for example, research on how the mind works, and the latter can be research on students' pre/misconceptions in the learning of algebra. A fourth source of knowledge (the wisdom of practice) is the knowledge that can be gleaned from the pedagogical principles that guide and are used by exemplary teachers. Those good practices can be a valuable source for teachers to gain significant pedagogical practices.

Richardson (1996) presented extensive discussions on the role of attitudes and beliefs in the education of teachers and learning to teach. Based on a comprehensive review of the related literature, Richardson identified three types of experiences that influence beliefs of future and in-service teachers, that in turn influence learning to teach and teaching. They are personal experiences, experiences with schooling and instruction, and experiences with formal knowledge. These influences were suggested by Ernest (1989) as well. According to Ernest (1989), while beliefs and attitudes are affective outcomes of teacher education, knowledge is a cognitive outcome, and knowledge is usually gained in teacher education through the content of instructional and learning experiences.

Buehl and Fives (2009) examined pre-service (n=53) and practising (n=57) teachers' beliefs about the source and stability of teaching knowledge; that is, where it comes from and if it changes. Adapting a grounded theory approach to data analyses, they identified six major themes, regarding the source of teaching knowledge, that refine and augment the previous investigations. These themes embrace a wide range of possible key sources of knowledge for teachers:

- Formal education (college coursework, workshops, conferences, subject area classes).
- Formal bodies of knowledge: information stores (books, literature, the internet); accumulated findings (educational research).
- Observational learning (formal or informal observations of good or bad teaching).

- Collaboration or interactions: meaning construction (co-construction of knowledge through sharing and collaborating); learning from others (e.g., experts, parents, peers, and colleagues).
- Enactive experiences: personal experiences (time spent in schools as a student, the way the individual was taught); professional experiences (on-the-job, actual teaching practice, listening to students).

Through a series of three interviews, Wilson et al. (2005) examined nine experienced mathematics teachers' views about good mathematics teaching and how good teaching develops. The teachers generally expressed that "good teaching requires a sound knowledge of mathematics, promotes mathematical understanding, engages and motivates students, and requires effective management skills". According to the teachers, "good teaching is developed from experience, [formal] education, personal reading and reflection, and interaction with colleagues" (p. 83).

The context of teacher education and development in Australia

In this section, I present an overview of the mathematics teacher education and development landscapes in Australia because teacher knowledge and specific beliefs about teaching and learning of mathematics can be influenced by their teaching contexts (e.g., Clark et al., 2014; Felbrich et al., 2012). Despite the obvious attention to both teacher quality standards and professional growth in Australia, some critical issues such as how teachers describe the knowledge base for teaching mathematics, and what they perceive as possible sources of that knowledge, have been remained unaddressed. This study concentrates on these critical yet unaddressed questions.

Teacher education

In Australia, initial teacher education is provided by universities; however, there is not one single or centralised pathway to become a mathematics teacher. Universities in all six states and two territories may offer different pathways. *In general*, secondary mathematics teachers have a Bachelor of Education (B.Ed.) with mathematics education specialisation, even though not all mathematics teachers in schools have this specialisation. A B.Ed. course (or degree) aims to equip teacher candidates with the necessary knowledge and skills to teach Years 7 to 12 mathematics, in government or non-government secondary schools. An admission requirement of these courses generally is demonstrating satisfactory examination performance in strong algebra, calculus, and statistics-based mathematics courses at the end of secondary school. These courses last four years, during which the teacher candidates are required to complete undergraduate mathematics units and mathematics education units that equip them how to teach secondary school mathematics from Years 7 to 12. The content of these units may vary from university to university. As part of their degree, teacher candidates complete school placements, and finally a nine-week placement before they graduate.

Teacher development

To attract, develop, recognise, and retain quality teachers, the Australian Institute for Teaching and School Leadership (AITSL) (2012a) has developed national, professional standards for teachers of all subjects including mathematics: the Australian Professional Standards for Teachers (the Standards). The Standards outline what teachers would know and be able to do to teach the subject effectively and are organised around three knowledge domains: professional knowledge, professional practice, and professional engagement (Table 2). Whilst not specific to teachers of mathematics, the Standards are aligned with those that are specific to mathematics teachers. That is, in 2006, the Australian Association of Mathematics Teachers [AAMT]—a national association of mathematics teachers—proposed a framework that describes the knowledge, skills, and attributes required for excellent teaching of mathematics. Its components are professional knowledge, professional attributes, and professional practice (Table 2). Expectations from teachers of mathematics in relation to each of these components have much in common with those of the Standards that apply to all subject area teachers.

Table 2. The standards set by the AITSL (2012a; the first row) and AAMT (2006; the second row)

Professional knowledge	**Professional engagement**	**Professional practices**
Students and how they learn The content and how to teach it	Professional learning Colleagues, parents/carers, and the community	Plan for and implement effective teaching and learning Create and maintain supportive and safe learning environments Assess, provide feedback and report on student learning
Professional knowledge	**Professional attributes**	**Professional practices**
Knowledge of students Knowledge of mathematics Knowledge of students' learning of mathematics	Personal attributes Personal professional development Community responsibilities	The learning environment Planning for learning Teaching in action Assessment

Teacher performance and development in schools are expected to be guided by the Australian Teacher Performance and Development Framework (AITSL, 2012b) which aims to promote a culture where professional conversations take place that can improve teaching and student outcomes. The Framework assumes that within this culture, teachers know what is expected of them, receive regular, useful feedback on their teaching (e.g., from peers, supervisor, or principal), and receive high-quality PL to improve their teaching. According to the Australian Charter for the Professional Learning of Teachers and School Leaders (AITSL, 2012c), which works hand in hand with the Framework, high-quality PL is relevant, collaborative, and future-focused (for more details, see AITSL, 2012c).

Methods

Research context and participants

Lower secondary teachers of mathematics who expressed interest in participating in a study group focusing on to support teachers' proficiency with algebra (hereafter the study group) served as informants for this study. The study group was advertised through the Mathematical Associations of Tasmania and six teachers from six different public schools located across Tasmania expressed interest. The participating teachers (PT) were predominantly teaching lower secondary mathematics (Years 7 to 10, 12-16 years old), with experience in teaching mathematics varying from 3 years to 16 years, and with different mathematics qualifications (Table 3).

Table 3. Demographic background of PTs (participating teachers)

	Level of Experience	Gender	Highest academic credential in mathematics	Grade levels taught
PT #1	8 years	Female	Statistics unit as part of Bachelor of Science degree	Years 7 and 8
PT #2	3 years	Female	Advanced Level Mathematics (Edexcel, UK-based).	Years 7 and 8
PT #3	12 years	Male	Calculus and Data/Statistics units as part of Bachelor of Agricultural Science degree	K-12, mostly Years 7-10
PT #4	3 years	Female	Bachelor of Primary Education with some mathematics units	Years 6-10
PT #5	16 years	Female	Bachelor of Education, mathematics minor	K-12
PT #6	6 years	Male	Bachelor of Education, mathematics minor	Years 7-10

Research instrument and data analysis

Before the study group meetings started, the teachers completed the open-ended questionnaire presented in Appendix A. The questionnaire was sent to them by email and returned to me within a week. The questionnaire aimed to access the PTs' perceptions of the types of professional knowledge that teachers of mathematics would have (Item 1) and how that knowledge is developed (Item 3), with a specific focus on how knowledge of students is gained (Item 4). Item 6 sought PTs' motivation to participate in the study group. Additionally, Items 7 and 8 intended to gain insight into their attitudes towards mathematics, and their views about the aims of mathematics education for students, respectively.

For the current investigation, I was particularly interested in perceptions of the PTs on the knowledge needed to teach mathematics and the potential sources of that knowledge. Whilst Items 4 and 5 addressed the PTs' perceptions of a specific dimension of teacher knowledge (knowledge of students) as part of the study group, this paper does not focus on a particular domain of teacher knowledge. Therefore, I focused my analysis on PTs' descriptions of the professional knowledge needed by mathematics teachers as a whole (including knowledge of students), and the potential sources of that knowledge, and analysed their responses to Items 1 to 6 holistically.

The key aspect of data analysis was the examination of the real words and language that the PTs used. Therefore, I conducted a content analysis (Stemler, 2001) which would enable me to discover and describe the perceptions of PTs on teacher knowledge and its sources. Out of the two approaches to coding the data (emergent versus a priori coding) (Stemler, 2001), I used a priori coding because the categories of teacher knowledge and its sources were based upon the theoretical or conceptual works in these fields. That is, I coded the statements made by the PTs according to the definitions or themes given by the scholars presented in Sections 2.2 and 2.3. While the coding unit was responses given to each of the questionnaire items (Item #1, Item #2, etc.), the recording unit was each of the sentences made or words used. Below are some example responses, and in brackets is the coding (italics are added).

> *How students learn* mathematics [knowledge of students], a good *understanding of the topics* [knowledge of content]. (PT #1, Item 1)
>
> *Learn from colleagues* [learn from others], *professional learning* opportunities [formal preparation] and *textbooks* [information stores]. (PT #1, Item 3)
>
> By *professional learning* in and outside of school [formal preparation] (PT #2, Item 3)
>
> By *formative assessment* [educational materials], *discussions with students* about their strategies [listening to students]. (PT #1, Item 4)

Once the coding was completed, the frequency of knowledge dimensions and sources of knowledge detected in the PTs' responses were counted and presented in Table 4.

In addition to perceptions of teacher knowledge and its sources, teachers' attitudes towards mathematics and views about overall goals of mathematics education are also important (Ernest, 1989; 2015), and they need to be considered in discussions on effective teaching of mathematics. In that sense, the PTs' responses to Items 7 and 8, where they described what mathematics mean to them and the aims of mathematics education, were analysed and presented to portray a comprehensive picture. In analysing responses to these items, I intended to capture the trends in their descriptions and implemented a summative content analysis (Hsieh & Shannon, 2005). The complete data set with the described data analysis approach applied to it are given in Appendix B (responses to Items 1 to 6) and Appendix C (responses to Item 7 and 8).

Results

PTs' perceptions of teacher knowledge

Teacher knowledge dimensions and potential sources of the knowledge for teachers revealed in the PTs' responses are shown in Tables 4 and 5. The PTs usually indicate the importance of teacher professional knowledge (Item 2, Appendix B) and put greater importance on understanding students' thinking and learning needs (Item 5, Appendix B). The ways they describe the kind of knowledge that a teacher of mathematics would have (Item 1) can be understood as being predominantly grounded in PCK, which is especially close to the PCK definitions of An et al. (2004) and Ball et al. (2008).

Table 4. The types of teacher knowledge revealed in PTs' responses (f: number of mentions)

The types of teacher knowledge (Item 1)	Motivations for the study group (Item 6)
Knowledge of content (f=2)	Knowledge of content and teaching (f=4)
Knowledge of content and teaching (f=1)	Knowledge of teaching (f=2)
Pedagogical content knowledge (f=1)	Knowledge of content (f=1)
Specialised content knowledge (f=1)	*Interactions or collaboration:
Knowledge of students (f=1)	learn from/with others (f=3)
Knowledge of pedagogy (f=1)	
*Interactions or collaboration:	
learn from others (f=1)	

*Sources of knowledge emerged in cross items

In responding to Item 1, PT #1 mentions foundational understanding of mathematics and the knowledge of how student learn mathematics. PT #2's statements clearly embrace the necessity for teachers to have specialised content knowledge identified by Ball et al. (2008). PT #2 emphasises the need for teachers to understand the sequence of mathematical understanding, know how to differentiate mathematical content (possibly) according to student needs, know how to represent mathematical ideas in different forms and solve mathematical problems in different ways. PT #5 puts emphasis on "how to teach mathematics" both when describing the types of professional knowledge for teachers (Item 1) and its importance (Item 2). According to her, if a teacher of mathematics does not have that knowledge, then for them access "to appropriate professional learning is a must" (PT #5, Item 2). The mentioned teaching knowledge includes knowing various pedagogies to work with students with different abilities and "basic content to start with, however, this will need to be built upon depending on the level of mathematics being taught" (PT #5, Item 1) which reflects the knowledge of content and teaching proposed by Ball and colleagues (2008). Among the remaining two teachers, in PT #3's response there is no specific mention of teacher knowledge but instead the role and importance of teacher interactions or collaboration is emphasized. PT #4 alludes to three teacher knowledge domains identified by scholars (e.g., Shulman, 1987): knowledge of content, PCK, and knowledge of pedagogy.

Although the PTs were not explicitly asked what the knowledge of students constitutes, I integrated their responses to Item 5, where they were asked for their opinion about in what way it is important for teachers to know their students' understanding of a particular mathematical content, with their responses to Item 1, to gain insight into what knowledge of students mean to them. While the statements vary from teacher to teacher, the overall descriptions of knowledge of students are quite similar. The PTs refer in some way to the knowledge of student cognition that has been described in relevant research literature (An et al., 2004; Fennema & Franke, 1992: Marks, 1990) which includes knowing students' thinking, learning, difficulties, and mistakes. PT #1, PT #3, PT #5, and PT #6 explicitly mention the role of knowing students' common misunderstandings. PT #6 additionally underlines the importance of understanding the different stages of content comprehension in students and planning for them (see Appendix B).

PTs' perceptions of sources of teacher knowledge

The ways PTs describe how teachers continue to enhance their professional knowledge (Item 3) can be understood as being grounded in three of Buehl and Fives's (2009) themes regarding the source of teaching knowledge: formal preparation, formal bodies of information, and interactions or collaboration with others (Table 5).

The PTs give considerable credit to PL opportunities (PT #1, PT #2, PT #4, PT #5, PT #6), including conferences, workshops, online courses, and webinars (PT #3, PT #5) and colleagues or peers (PT #1, PT #3, PT #4, PT #5, PT #6) as potential sources for enabling the development of knowledge needed for teaching mathematics. In addition to these, PT #1 and PT #4 indicate that teachers could continue to develop their professional knowledge through textbooks (PT #1) or consulting research or other resources (PT #4). According to PT #3, PT #5, and PT #6, knowledge can be co-constructed by means of participating in PL teams in one's own school and beyond (PT #3 and PT #6) or in moderation meetings (PT #5) where usually teachers with different experiences and from across a range of year levels review student work samples against curriculum achievement standards.

Table 5. Sources of teacher knowledge revealed in PTs' responses (f: number of mentions)

Sources of teacher knowledge (Item 3)	**Sources of knowledge of students (Item 4)**
Formal preparation (f=8)	Formal bodies of information:
Formal bodies of information:	assessment materials (f=5)
textbooks (f=1)	test and testing materials (f=2)
educational research (f=1)	Professional experiences:
Interactions or collaboration:	listening to students (f=4)
learn from others (f=4)	on-the-job (f=2)
meaning construction (f=3)	

The observed sources of knowledge in PTs' responses to Item 4 differ from those detected in response to Item 3 and include Buehl and Fives's (2009) professional experiences theme. That is, according to the PTs, student thinking can be known through listening to students (PT #1, PT #2, PT #4, PT #5) or through enactive experiences, i.e., on-the-job (PT #4, PT #6). As PT #6 describes clearly, teachers know about their students' understanding of a particular mathematical content (Item 4):

> Through assessment and *observation*, really. Like, they *emerge through their work samples*, the *errors they make*, and the *questions they ask*.

Another source of knowledge of students in the PTs' responses is Shulman's (1987) educational materials and institutional contexts or Buehl and Fives' (2009) formal bodies of information. Gaining knowledge of student understanding through conducting formative (PT #1, PT #2, PT #5, PT #6) or diagnostic (PT #5) assessments, or using test or testing materials (PT #4, PT #5), are among the sources that constitute this category.

Slightly differently, PT #4 alludes that [teachers can know about their students' understanding of a mathematical content] "*by knowing* possible student misconceptions of the topic …" (Item 4, Appendix B). In a sense, this statement reflects Shulman's (1987) *research on schooling* or Buehl and Fives' (2009) *educational research* as a source from where teachers may acquire student misconceptions of a particular mathematical content. However, I chose not to code this statement as whether those misconceptions are known from the research or elsewhere is unclear. Nonetheless, there is some comfort that PT #4 may be aware of this source of knowledge.

Motivations for the study group

PTs had different motivations for participating in the study group. Their responses to Item 6 show that their primary need was gaining professional knowledge—mostly knowledge of teaching algebra or knowledge of teaching (see Table 4). PT #6 was "excited to participate because [he] want[s] to grow as a teacher".

The ways four of the PTs describe knowledge of teaching (PT #1, PT #3, PT #4, PT #5, Appendix B) can be understood as being grounded in Ernest's (1989) pedagogical knowledge of mathematics or Ball et al.'s (2008) knowledge of content and teaching. That is, through participating in the study group, PT #1, PT #3, and PT #5 hoped to gain knowledge on how to teach algebra effectively. PT #5 hoped to "expand [her] own understanding of algebra and the various methods to teaching this subject". PT #4 wished to "extend [her] understanding of teaching problem solving" and expected to know different ways of representing, explaining and teaching problem-solving. Conversely, PT #2 and PT #6 sound to be expecting to improve their generic teaching techniques.

Some PTs found the study group experience to be an opportunity for interacting or collaborating with other teachers (Table 4, marked with asterisk); that is, obtaining different perspectives (PT #2), sharing their ideas and those of their peers (PT #6), or learning "with like-minded people" (PT #5). This once again indicates that some of the PTs consider interactions or collaboration with peers as a major source of acquiring or developing professional knowledge.

Attitudes towards mathematics and views about goals of mathematics education

The PTs' statements to complete the prompt: 'To me mathematics is …' (Item 8, Appendix C) reveal that they have positive attitudes towards mathematics. To PT #3, mathematics is "a challenge but rewarding and beautiful", to PT #4 mathematics is "an enjoyable puzzle to be solved", and to PT #2, it is about "understanding logic and procedures". The statements of PT #1 and PT #5 show that they put considerable value on mathematics. According to them, mathematics is: "an engaging and challenging way to understand relationships in our Universe" (PT #1) or "an integral part of life which is much more interesting and engaging than a lot of people give it credit for" (PT #5). PT #6 expresses a slightly different orientation (italics are added):

> [To me, mathematics is] … *learnable*. Anyone can learn it. And the more you learn it, *the more connections you [students] make, and the more it unfolds and reveals itself*. (PT #6)

Their responses to Item 7 (see Appendix C) reveal that the PTs' views about the aims of mathematics education for students are mainly related to building confidence in students (PT #1, PT #3, PT #5), and using mathematical skills in their everyday lives (PT #1, PT #2). To PT #5, mathematics education additionally enables students to see how mathematics is useful in real life situations, and to PT #4, mathematics provides students with lifelong skills (reasoning and justification), which are needed in most aspects of life, and skills enabling them to problem-solve in the real world.

Discussion

The discussion aims to interpret the major findings revealed in the PTs' responses organised into three sections.

Perceptions of teacher knowledge

Findings have revealed that the PTs in this study gave importance to teacher professional knowledge (Mosvold & Fauskanger, 2013, 2014) and foregrounded the understanding of student thinking for catering to students' learning needs (Perry, 2007). To some extent, they did recognise the knowledge bases needed to teach mathematics identified by scholars in the literature. As outlined in Table 6, these included knowledge of content, knowledge of content and teaching, PCK, specialised content knowledge, knowledge of students, and knowledge of pedagogy. Overall, however, PTs' perceptions of the knowledge needed for teaching mathematics seemed to be relatively narrow. In studies on teaching and teacher education, content knowledge has been considered as one of the key aspects of teacher knowledge and 'good' teaching (e.g., Charalambous, 2015; Fives & Buehl, 2008; Leong, 2015; Wilson et al., 2005). In PTs' descriptions, there were comparatively fewer references to content knowledge. Several aspects of teacher knowledge, including knowledge of mathematical horizon, knowledge of curriculum and materials, were not observed. More importantly, usually just a single knowledge dimension was articulated.

Table 6. The teacher knowledge and its sources revealed in PTs' responses (f: number of mentions)

Domains of teacher knowledge	Sources of teacher knowledge
Knowledge of content and teaching (f=5)	Formal preparation (f=8)
Knowledge of content (f=3)	Formal bodies of information
Knowledge of teaching (f=2)	assessment materials (f=5)
Pedagogical content knowledge (f=1)	test and testing materials (f=2)
Specialised content knowledge (f=1)	textbooks (f=1)
Knowledge of students (f=1)	educational research (f=1)
Knowledge of pedagogy (f=1)	Interactions or collaboration
	learn from others (f=8)
	meaning construction (f=3)
	Professional experiences
	listening to students (f=4)
	on-the-job (f=2)

It is well recognised that teaching is a complex task (Fives & Buehl, 2008), and teacher knowledge is multidimensional (see also Mosvold & Fauskanger, 2013). The perceptions revealed by the PTs' responses on the knowledge needed by mathematics teachers may be contextual (Clark et al., 2014; Kaiser & Vollstedt, 2007; Mosvold & Fauskanger, 2013), and indeed it points to the fact that teacher education contexts play a role in teachers' perceptions of teaching or teacher knowledge (Felbrich et al., 2012; Werler & Tahirsylaj, 2022). As noted earlier, not all mathematics teachers in schools in Australia have a specialisation in mathematics or mathematics education (see Section 2.4; and also, Weldon, 2016). It is likely that some teachers in this study did not study mathematics education units other than teacher education units. It might be that they were, and are, unaware of the specific knowledge base that is unique to teachers of mathematics.

Given that teachers' perceptions of knowledge, as part of their thought process (Ernest, 1989; Richardson, 1996), may influence how and what they acquire from teacher education or professional development programs (Fives & Buehl, 2008; Ponte, 2011), Australian mathematics teacher education and development communities need to devote more attention to this issue. If teachers are unaware of knowledge needed for teaching mathematics, or consider that knowledge to be unidimensional, it is likely that their PL intentions or choices would reflect this (Richardson, 1996). The notion of teacher knowledge could be addressed in teacher education and professional development programs more intentionally, if there is a genuine desire to increase the number of quality mathematics teachers who have the necessary mathematical understanding and skills. When teachers are introduced to the concept of teacher knowledge that is unique to and needed for mathematics teaching, its aspects (e.g., knowledge of student thinking in algebra learning), and constituents of each aspect (e.g., knowing students' preconceptions in algebra), they may become more aware of this concept and make informed decisions in their PL efforts aiming to upgrade their mathematical knowledge and skills (Fives & Buehl, 2008).

Perceptions of sources of teacher knowledge

Various sources of knowledge for teachers identified in the relevant research literature were evident in the PTs' responses (see Table 6). These included both formal and informal knowledge sources. Among the former, categories were formal preparation (workshops, conferences) and formal bodies of information (textbooks, educational research, assessment, testing), while among the latter, they were interactions or collaboration (learning from colleagues, constructing meaning) and professional experiences (listening to students, on-the-job). The formal preparation theme included the professional learning opportunities subtheme such as online courses, workshops, conferences, and PL teams in schools. It is worth noting that university coursework programs (e.g., bachelor or postgraduate programs) as formal preparation was not detected. This once again points to the fact that educational contexts play a role in the perceptions of teachers (Felbrich et al., 2012; Mosvold & Fauskanger, 2013). In Australia, while they may undertake intensive in-service programs, generally, it is unusual for teachers to pursue postgraduate studies (Perry, 2007). As to undergraduate university coursework programs, PTs' relevant experiences might be limited as their major university training was in fields other than mathematics education, or it might be that university coursework programs are in the distant past for practising teachers, as opposed to some pre-service teacher (Hatisaru & Collins, 2023).

The emergence of some categories (namely, professional experiences; and assessment, test and testing materials) only in the responses to Item 4 where PTs articulated on knowledge of students shows that, consistent with Buehl and Fives (2009), PTs perceived that different types of knowledge come from different sources. That is, for them, whilst teacher knowledge as a whole may be gained or enhanced through PL events and interactions with colleagues, knowledge of students comes from their own practice—i.e., through listening, observing, or conferencing with students.

In PTs' responses, there were particularly strong references to interacting with others and learning from/with them (Buehl & Fives, 2009; Wilson et al., 2005). This inclination suggests a collaborative approach to gaining or developing knowledge and reflects the AITSL documents' emphasis on teacher performance and development in schools (AITSL, 2012b; 2012c). There are other sources of knowledge that may inform teachers as well. For example, there are abundant research studies on many of the relevant teacher knowledge domains. A specific research literature underpins knowledge of student cognition (Fennema & Franke, 1992; Kazemi & Franke, 2004; Özgün-Koca et al., 2020), and that literature has been growing. The acquisition of knowledge of students may largely occur in practice, but also may be constructed or developed through consulting the educational research (AITSL, 2012a; Buehl & Fives, 2009; Fennema & Franke, 1992; Kazemi & Franke, 2004). In the PTs' descriptions, there was almost no reference to the research literature as a source of teacher knowledge. Despite their attention to PL opportunities, there remains the question of to what extent PTs viewed educational research as a source of teacher knowledge, and this is an area that needs further investigations.

Attitudes towards mathematics

As opposed to primary pre-service teachers who sometimes hold negative attitudes towards mathematics (see Cady & Rearden, 2007; Maasepp & Bobis, 2014), PTs in this study expressed positive attitudes towards mathematics and put value on it. It is important to note that the PTs in this study were motivated secondary teachers who expressed interest in participating in a study group to enhance their proficiency with algebra. It would be reasonable to expect that they were established teachers in their profession and had positive dispositions about mathematics. Therefore, it was not surprising that they expressed positive sentiments about mathematics, like the group of selected teachers in Perry's (2007) study.

Views about goals of mathematics education

The PTs' views about the aims of mathematics education for students were mainly related to building confidence in students and utilising mathematical skills in their everyday lives. The teachers' responses conveyed a sense of Ernest's (2015) functional numeracy aspect of mathematics education, and this result corresponds with findings by Kaiser and Vollstedt (2007) who reported that Australian teachers hold a functional view of mathematics where mathematics is viewed as a tool used for practical purposes (e.g., describing or explaining physical phenomena). PT #1, PT #3, and PT #5 also foregrounded building confidence in students within mathematics education and their words tentatively reflect Ernest's (2015) visionary goals of mathematics education that includes mathematical confidence, mathematical problem-solving, and appreciation of mathematics components (for more detail see Ernest, 2015 and Hatisaru, 2020b). A reasonable question is to what ways the PTs' understanding was consistent with that of Ernest (2015) regarding mathematical confidence, which refers to being confident in learning and using mathematics, persistence and willingness in solving mathematical tasks. Answering this question requires future research utilising observations or interviews.

Implications for teacher education and development

Two of the major findings presented above are relevant when considering the implications of this research for teacher education and development. First, the introduction of the Standards (AITSL, 2012a), along with its accompanying documents (AITSL, 2012b, 2012c), clearly requires teachers to hold knowledge and skills for excellence in teaching. These standards include both professional knowledge and professional engagement and practices. They have much in common with those set by the AAMT (2006) specific to mathematics teachers (see Section 2.4). If teacher education programs are to graduate teachers of mathematics who create excellence in teaching mathematics, it is necessary that attention is given to the content-related knowledge domains that have been communicated both in the AITSL and AAMT documents. The design of teacher education programs needs to consider making those knowledge domains and their constituents explicit to future teachers of mathematics.

Second, the PTs' words in Item 6 where they reflected on their motivations for participating in the study group provide some genuine, valuable insights into the possible needs of teachers in teaching algebra. Their responses have shown that the PTs would hope to gain content-related knowledge to strengthen their professional capability. The knowledge domains evident in their responses were knowledge of content and teaching, knowledge of teaching, and knowledge of content. Perhaps PT #1 echoed the need for content-specific knowledge best when she wrote:

> I hope to learn effective ways to teach Algebra, particularly for Grade 8. I'm not a trained Maths teacher so looking for creative and effective ways to teach so that students learn and feel comfortable with the subject (Item 6, PT #1, Appendix B).

As mentioned by PT #1, and as noted earlier in this paper and in Perry (2007) and Weldon (2016), not all mathematics teachers in schools in Australia are specialist mathematics teachers. Whilst a great amount of PL opportunities (both state-based and national) is available for teachers in general and teachers of mathematics in particular, to my observation, not all of them intentionally target the upgrading of teachers' knowledge and skills of specific mathematical content or of particular knowledge domains (see also Du Plessis, 2020). Studies in this field show that the most effective PL programs focus on content-related knowledge that teachers can directly use during daily instruction including content knowledge, knowledge of students, and curriculum materials (Hill et al., 2020; Kazemi & Franke, 2004; Özgün-Koca et al., 2020). It is important that PL initiatives are intentional and consider the development of needed knowledge in teachers. The theoretical and conceptual models of teacher knowledge presented in this paper can provide a framework for considering some of the questions which the construction of such teacher education programs, or teacher development, raises.

Limitations of the study

The study had some limitations associated with its methods, and I employed several strategies to reduce them (Healy & Perry, 2000; Noble & Smith, 2015). Firstly, the data were generated and analysed by the author, who might hold specific biases. I attempted to reduce that limitation by presenting an extensive review of the theoretical and conceptual perspectives on teacher knowledge, and its sources, and built the data generation instrument upon relevant research studies (see Appendix A). In addition to detailed, clear and transparent descriptions of the research process, the data set with the data analysis approach was appended to the paper (Appendices B and C). Secondly, whilst an open-ended questionnaire was an efficient method for generating rich, textual data (Jackson & Trochim, 2002) giving access into perceptions of teachers, the resulting data were limited to those identified in responses to given eight items. To obtain a comprehensive understanding of PTs' perceptions, I analysed and presented their responses to all eight items, independent of each item's individual focus. That not only contributed to obtaining a broad and detailed picture of PTs' perceptions, but also to triangulating the PTs' articulations across items. Findings of the questionnaire have indicated that the instrument detects teachers' perceptions. This is promising, as open-ended questionnaires could be used with larger samples. It is, nonetheless, acknowledged that PTs' responses to questionnaire items might not reflect their perceptions entirely; interviews or classroom observations might have generated additional, valuable insights. The given descriptions in this paper, therefore, might be a snapshot of this scenario.

Finally, the sample of study was restricted to six lower secondary teachers of mathematics teaching in a single state in Australia, who showed interest in participating in a teacher study group. The findings of the study may not be broadly applicable to different contexts in Australia, and/or elsewhere. An important next step is to investigate teachers' perceptions of teacher knowledge from broader Australian and global contexts.

Conclusion

A diversity of perceptions was identified amongst the teachers. Across several knowledge domains, ten teachers mentioned knowledge of content and teaching, knowledge of content, or knowledge of teaching, and four recalled pedagogical content knowledge, specialised content knowledge, knowledge of students, or knowledge of pedagogy. The elaboration of the sources of knowledge highlighted that the teachers perceived formal preparation, formal bodies of information, interactions or collaboration, or professional experiences as sources of knowledge for teachers (see Table 6). Further research in regard to how these perceptions contribute to the ways in which teachers acquire the necessary knowledge that is needed for teaching mathematics, and how they inform teachers' professional learning decisions, is warranted.

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References

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education, 7*, 145–172. https://doi.org/10.1023/B:JMTE.0000021943.35739.1c

Australian Association of Mathematics Teachers (AAMT). (2006). *Standards for excellence in teaching mathematics in Australian schools.* The Australian Association of Mathematics Teachers Inc.

Australian Institute for Teaching and School Leadership. (2012a). *The Australian Professional Standards for Teachers.* AITSL.

Australian Institute for Teaching and School Leadership. (2012b). *Australian Teacher Performance and Development Framework*. AITSL.

Australian Institute for Teaching and School Leadership. (2012c). *Australian Charter for the Professional Learning of Teachers and School Leaders.* AITSL.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal for Teacher Education, 59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Buehl, M. M., & Fives, H. (2009). Exploring teachers' beliefs about teaching knowledge: Where does it come from? Does it change?. *Journal of Experimental Education, 77,* 367–408. https://doi.org/10.3200/JEXE.77.4.367-408

Bryan, C. A., Wang, T., Perry, B., Wong, N-Y., & Cai, J. (2007). Comparison and contrast: similarities and differences of teachers' views of effective mathematics teaching and learning from four regions. *ZDM Mathematics Education, 39*(4), 329–340. https://doi.org/10.1007/s11858-007-0035-2

Cady, J. A., & Rearden, K. (2007). Pre-service teachers' beliefs about knowledge, mathematics, and science. *School Science and Mathematics*, *107*(6) 237–245. https://doi.org/10.1111/j.1949-8594.2007.tb18285.x

Charalambous, C. Y. (2015). Working at the intersection of teacher knowledge, productive dispositions, and teaching practice: A multiple-case study. *Journal of Mathematics Teacher Education, 18*(5), 427–445. https://doi.org/10.1007/s10857-015-9318-7

Charalambous, C. Y., & Pitta-Pantazi, D. (2016). Perspectives on priority mathematics education. Unpacking and understanding a complex relationship linking teacher knowledge, teaching, and learning. Taylor & Francis. In L. D. English, & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd edition). Taylor & Francis.

Clark, L. M., DePiper, J. N., Frank, T. J., Nishio, M., Campbell, P. F., Smith, T. M., Griffin, M. J., Rust, A. H., Conant, D. L., & Choi, Y. (2014). Teacher characteristics associated with mathematics teachers' beliefs and awareness of their students' mathematical dispositions. *Journal for Research in Mathematics Education, 45*(2), 246–284. https://doi.org/10.5951/jresematheduc.45.2.0246

Du Plessis, A. E. (2020). The lived experience of out-of-field STEM teachers: A quandary for strategising quality teaching in STEM?. *Research in Science Education*, *50*, 1465–1499. https://doi.org/10.1007/s11165-018-9740-9

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching*, *15*(1), 13–33.

Ernest, P. (2015). The social outcomes of learning mathematics: Standard, unintended or visionary? *International Journal of Education in Mathematics, Science and Technology, 3*(3), 187–192. https://eric.ed.gov/?id=EJ1066357

Felbrich, A., Kaiser, G., & Schmotz, C. (2012). The cultural dimension of beliefs: An investigation of future primary teachers' epistemological beliefs concerning the nature of mathematics in 15 countries. *ZDM Mathematics Education, 44*(3), 355–366. https://doi.org/10.1007/s11858-012-0418-x

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning,* (pp. 147–164). Macmillan.

Fives, H., & Buehl, M. M. (2008). What do teachers believe? Developing a framework for examining beliefs about teachers' knowledge and ability. *Contemporary Educational Psychology, 33*(2), 134–176. https://doi.org/10.1016/j.cedpsych.2008.01.001

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education.* Teachers College Press. https://lccn.loc.gov/90038509

Handal, B. (2003). Teachers' mathematical beliefs: A Review. *The Mathematics Educator, 13*(2), 47–57. https://openjournals.libs.uga.edu/tme/article/view/1863

Hatisaru, V. (2020a). Exploring evidence of mathematical tasks and representations in the drawings of middle school students. *International Electronic Journal of Mathematics Education*, *15*(3), 1–21.

Hatisaru, V. (2020b). Perceived need for mathematics among lower secondary students. *Australian Mathematics Education Journal, 2*(1), 9–14.

Hatisaru, V., & Collins, J. (2023). Preservice secondary mathematics teachers' perceptions of teacher knowledge and its sources. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia (pp. 251–258). Newcastle: MERGA.

Hatisaru, V., Falloon, G., Seen, A., Fraser, S., Powling, M., & Beswick, K. (2023). Educational leaders' perceptions of STEM education revealed by their drawings and texts. *International Journal of*

Mathematical Education in Science and Technology, *54*(8), 1437-1457. https://www.tandfonline.com/doi/full/10.1080/0020739X.2023.2170290

Healy, M., & Perry, C. (2000). Comprehensive criteria to judge validity and reliability of qualitative research within the realism paradigm. *Qualitative Market Research, 3*(3), 118–126. https://doi.org/10.1108/13522750010333861

Hill, H. C., Lynch, K., Gonzalez, K. E., & Pollard, C. (2020). Professional development that improves STEM outcomes. *Phi Delta Kappan, 101*(5), 50–56. https://doi.org/10.1177/0031721720903829

Hsieh, H. F., & Shannon, S. E. (2005). Three approaches to qualitative content analysis. *Qualitative Health Research, 15*(9), 1277–1288. https://doi.org/10.1177/1049732305276687

Jackson, K. M., & Trochim, W. M. K. (2002). Concept mapping as an alternative approach for the analysis of open-ended survey responses. *Organization Research Method, 5,* 307–336. https://doi.org/10.1177/109442802237114

Jong, C., & Hodges, T. E. (2015). Assessing attitudes toward mathematics across teacher education contexts. *Journal of Mathematics Teacher Education, 18,* 407–425. https://doi.org/10.1007/s10857-015-9319-6

Kaiser, G., & Vollstedt, M. (2007). Teachers' views on effective mathematics teaching: Commentaries from a European perspective. *ZDM Mathematics Education, 39,* 341–348. https://doi.org/10.1007/s11858-007-0036-1

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7,* 203–235. https://doi.org/10.1023/B:JMTE.0000033084.26326.19

Leong, K. E. (2015). What are the important attributes of good mathematics teaching? *Asia-Pacific Educational Researcher, 24*(1), 211–223. https://doi.org/10.1007/s40299-014-0173-6

Maasepp, B., & Bobis, J. (2014). Prospective primary teachers' beliefs about mathematics. *Mathematics Teacher Education and Development, 16*(2), 89–107.

Marks, R. (1990). Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Teacher Education, 41*(3), 3–11. https://doi.org/10.1177/002248719004100302

Mosvold, R., & Fauskanger, J. (2013). Teachers' beliefs about mathematical knowledge for teaching definitions. International *Electronic Journal of Mathematics Education, 8*(2-3), 43–61. https://doi.org/10.29333/iejme/273

Mosvold, R., & Fauskanger, J. (2014). Teachers' beliefs about mathematical horizon content knowledge. *International Journal for Mathematics Teaching and Learning,* 1–16. https://www.cimt.org.uk/journal/mosvold2.pdf

Noble, H., & Smith, J. (2015). Issues of validity and reliability in qualitative research. *Evidence-Based Nursing, 18,* 34–35. http://dx.doi.org/10.1136/eb-2015-102054

Özgün-Koca, S. A., Lewis, J. M., & Edwards, T. (2020). Fostering middle school teachers' mathematical knowledge for teaching via analysis of tasks and student work. *Mathematics Teacher Educator, 9*(1), 50–62. https://www.jstor.org/stable/10.5951/mte.2020.0004

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307–332.

Perry, B. (2007). Australian teachers' views of effective mathematics teaching and learning. *ZDM Mathematics Education, 39*(4), 271–286. https://doi.org/10.1007/s11858-007-0032-5

Ponte, J. P. (2011). Teachers' knowledge, practice, and identity: Essential aspects of teachers' learning. *Journal of Mathematics Teacher Education, 14*(6), 413–417. https://doi.org/10.1007/s10857-011-9195-7

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of research on teacher education* (pp. 102–119). Macmillan.

Schoen, R. C., & LaVenia, M. (2019). Teacher beliefs about mathematics teaching and learning: Identifying and clarifying three constructs. *Cogent Education, 6*(1). https://doi.org/10.1080/2331186X.2019.1599488

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1–23. https://doi.org/10.17763/haer.57.1.j463w79r56455411

Stemler, S. (2001). An overview of content analysis. *Practical Assessment, Research, and Evaluation, 7*. Article 17. https://doi.org/10.7275/z6fm-2e34

Viholainen, A., Asikainen, M., & Hirvonen, P. E. (2014). Mathematics student teachers' epistemological beliefs about the nature of mathematics and the goals of mathematics teaching and learning in the beginning of their studies. *Eurasia Journal of Mathematics, Science and Technology Education, 10*(2), 159–171. https://doi.org/10.12973/eurasia.2014.1028a

Weldon, P. R. (2016). *Out-of-field teaching in Australian secondary schools*. Australian Council for Educational Research. https://research.acer.edu.au/cgi/viewcontent.cgi?article=1005&context=policyinsights

Werler, T. C., & Tahirsylaj, A. (2022). Differences in teacher education programmes and their outcomes across Didaktik and curriculum traditions. *European Journal of Teacher Education, 45*(2), 154–172. https://doi.org/10.1080/02619768.2020.1827388

Wilson, P. S., Cooney, T. J., & Stinson, D.W. (2005). What constitutes good mathematics teaching and how it develops: Nine high school teachers' perspectives. *Journal of Mathematics Teacher Education, 8,* 83–111. https://doi.org/10.1007/s10857-005-4796-7

Appendix A. The questionnaire used in the study

The purpose of this open-ended questionnaire is to have an opportunity to know a bit about your mathematics teaching background, views on mathematics, mathematics education, and knowledge needed for teaching it. There is not a best single response to the items. I hope you will feel comfortable writing me what you think because I am interested in knowing about your views on the relevant issues before starting our workgroup.

Personal data

- Level of experience:
- Gender:
- Highest academic credential in mathematics:
- Grade levels taught:

Views about mathematics education and teacher knowledge

(1) What kind of professional knowledge should a teacher of mathematics have?
(2) How important is it for teachers to have this knowledge?
(3) How do teachers continue to enhance their professional knowledge?
(4) How do teachers know about their students' strategies and understanding of a particular mathematical content?
(5) In what way is it important for teachers to know their students' approach and understanding of a particular mathematical content?
(6) There might be different reasons that teachers choose to participate in professional learning events. Why did you decide to participate in the workgroups? For instance, what are your hopes/concerns for the workgroup?
(7) From your point of view, what are the goals of mathematics education for students?
(8) Please complete the sentence:
To me, mathematics is …

* Items 1 to 5 and Item 7 were adapted from An et al. (2006), and Item 8 comes from Sam and Ernest (2000).

*** References**

An, S., Kulm, G., Wu, Z., Ma, F., & Wang, L. (2006). The impact of cultural differences on middle school mathematics teachers' beliefs in the U.S. and China. In F. K. S. Leung, K. D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics education in different cultural traditions: A comparative study of East Asia and the West* (pp. 449–465). Springer.

Sam, L. C., & Ernest, P. (2000). A survey of public images of mathematics. *Research in Mathematics Education*, *2*(1), 193–206. https://doi.org/10.1080/14794800008520076

Appendix B. Participating teacher (PT) responses to Items 1 to 6 (italics added) and their mapping to teacher knowledge domains and sources [in parenthesis]

	The content of teacher knowledge (Item 1)	Its importance (Item 2)	Sources of teacher knowledge (Item 3)	Sources of teacher knowledge of students (Item 4)	Its importance (Item 5)	Motivations for the Algebra PL (Item 6)
PT #1	*How students learn mathematics* [knowledge of students], a good *understanding of the topics* [knowledge of content].	Quite important I believe.	*Learn from colleagues* [learn from others], *professional learning opportunities* [formal preparation] and *textbooks* [information stores].	By *formative assessment* [educational materials], *discussions with students* about their strategies [listening to students].	So, they know *common misunderstandings* and can help to correct them and build on their current knowledge.	I hope to *learn effective ways to teach Algebra*, particularly for Grade 8. I'm not a trained Maths teacher so looking for *creative and effective ways to teach* so that *students learn and feel comfortable with the subject* [knowledge of content and teaching].
PT #2	*How to differentiate*. Understand *the continuum of mathematical understanding*, so that they *can differentiate tasks as required*. Know *various different ways to represent and solve mathe-matical tasks.* [specialised content knowledge]	Extremely important.	By *professional learning* in and outside of school [formal preparation]	By *asking the student questions* and asking them *to show their workings and answers* [listening to students].	If they do not understand the *students' thinking*, they cannot guide them to the next step.	To *collaborate with other teachers* and obtain better perspective [learn from others]. To know *different ways of problem-solving and representation* [content knowledge] For *improvement of* my *teaching techniques* [knowledge of teaching].
PT #3	I feel it is important to keep connected and share materials. *Learn of each other* and see current practice [learn from others].	It is important to *build and maintain professional networks* to keep our *teaching practice effective* and meaningful.	Through *conferences* [formal preparation] and *professional learning teams* at their workplace [meaning construction], doing *online courses* [formal preparation], *learning off each other* [learn from others].	*Through* an alignment of skills linked to the ACF via achievement *standard rubrics that we have created* [educational materials]. We create common assessment tasks that align to the rubric.	Important to see if there are any common *misunderstandings* that you can address and find powerful and effective methods to change their thinking.	Build some understanding around the *teaching of algebra using rich tasks* [knowledge of content and teaching].
PT #4	Knowledge of *Content*, *Pedagogical Content Knowledge*, Knowledge of *Pedagogy* [three teacher	Extremely	*Talking to other teachers* [learn from others], *research of their own* [accumulated findings],	By knowing possible *student misconceptions of the topic* [Not coded]. From *pre- and post-testing* [test and testing materials]. By getting to know	Very important. You need to be able to teach from *where the student is currently at* and build on this understanding.	I am continually trying to improve my practice and I saw this as a good opportunity to *extend my understanding of teaching problem solving*. I

	knowledge domains identified in the literature]		*professional learning* [formal preparation]	their students. *Conferencing and class discussions* [on-the-job]. *Asking students to justify their answers* and *show all working out* [listening to students].		hope *to find different ways of explaining/teaching/ represen-ting problem solving concepts*. [knowledge of content and teaching]
PT #5	Multiple *pedagogies and approaches* to work from depending on abilities and confidence of students. Basic content to start with, however, this will need to be *built upon depending on the level of mathematics being taught* [knowledge of content and teaching].	It is essential that teachers have this knowledge, particularly around *how to teach mathematics*. If they don't have it then access to appropriate professional learning is a must.	*Attendance at professional learning* both within the school and through external providers [formal preparation] *Moderation opportunities* [meaning construction] *Peer mentoring* [learn from others] Attendance at mathematics *conferences, workshops, online webinars* etc. [formal preparation]	*Formative assessment* [educational materials] Conducting *diagnostic assessments* (we us Assessment for Common Misunderstandings, and Scaffolding Numeracy in the Middle Years) [educational materials] *Standardised assessments* (PAT-M and NAPLAN) [test and testing materials] *Discussion with students about their thinking and reasoning* [listening to students] Conversations with previous teachers of said student/s	To better cater for *student learning needs* and allow access to concepts/lessons To identify *common misunderstandings* in their mathematical understanding to allow for appropriate support/intervention	To continue *to expand my own understanding of algebra* and the various methods of *teaching this subject* [knowledge of content and teaching]. *Professional learning with like-minded people*. [learn with others]
PT #6	No response given.	No response given.	Teachers enhance their professional knowledge *through professional learning* [formal preparation] and through *working as a community* [meaning construction]. This can be within schools (grade and subject teams) or within the broader maths teaching community.	Through *assessment* [educational materials] and *observation*, really. Like, they *emerge through their work samples*, the *errors they make*, and the *questions they ask* [on-the-job].	It's important for teachers to be able to anticipate *common errors* and consider them in their planning. It's important for teachers to understand *the different stages of comprehension* and plan for them. An example that sticks out to me is something I encountered on a PL day, that of different stages of multiplicative thinking.	I am excited to participate because I want to *grow as a teacher*. I'm looking forward to *sharing my ideas and those of my peers* and having this experience influence *how I teach my classes* [learn from others] [knowledge of teaching] and operate as a member of a high school teaching team.

Appendix C. Participating teacher (PT) responses to Items 7 and 8 (italics added)

	Aims of mathematics education for students (Item 7)	**To me, mathematics is … (Item 8)**
PT #1	For students *to feel confident in their abilities*, to connect ideas from multiple disciplines, to *be able to apply skills to their own lives* now and into the future.	an engaging and challenging way to understand relationships in our Universe.
PT #2	To be able *to be numerate*. To *have numeracy skills necessary for basic living*. To have *basic foundation of numbers*, patterns, generalisations. To have *computational skills*. To have capacity to follow logic.	understanding logic and procedures.
PT #3	To *improve confidence* in students.	a challenge, but rewarding and beautiful
PT #4	To have a good understanding of the fundamentals of maths and to *be able to problem solve in the real world*. To *be able to reason and justify their thinking* as these are *skills necessary in all aspects of life*.	an enjoyable puzzle to be solved.
PT #5	Building strategies and skills to use across mathematics. *Building confidence and resilience with mathematics*. Giving students a clear understanding of *how mathematics*, including more abstract concepts, *is useful in real life* situations.	an integral part of life which is much more interesting and engaging than a lot of people give it credit for.
PT #6	No response given.	… learnable. Anyone can learn it. And the more you learn it, the more connections you make, and the more it unfolds and reveals itself.