How do students communicate in writing and which difficulties do they have in solving a mathematical problem?

Como os alunos comunicam por escrito e quais as dificuldades que apresentam na resolução de um problema matemático?

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Abstract. By the end of compulsory education, students should have developed problem-solving and written communication skills. The Portuguese curriculum emphasises these skills by encouraging students to justify their ideas and decisions and to link their arguments in an organised and coherent way. Difficulties are an integral part of this process, perceived by students when solving problems or writing their solutions. Their identification is a necessary step so that they are overcome later. Therefore, our aims are: (1) to understand how students communicate their problem-solving in writing, and (2) to identify the difficulties they experience in problem-solving and written communication. We developed a system of categories to characterise written communication and to analyse the difficulties observed in students' resolutions. The resolutions analysed were written by six groups of 11th grade students in response to a mathematical problem presented to them. We obtained one incorrect, one partially correct and four correct resolutions, with diverse levels of justification, ranging from null to high, and with all types of justification that could be found. One of the groups did not present any difficulties, while the others had varying difficulties related to persistence, interpretation, selection of information, strategy, writing, and coherence.
Keywords: written communication; problem solving; difficulties; high school.

Resumo. Ao terminar a sua escolaridade obrigatória, é aconselhável que os alunos tenham desenvolvido capacidades de resolução de problemas e comunicação escrita. No currículo português, estas capacidades são realçadas devido à importância atribuída ao facto de os alunos serem incentivados a justificar as suas ideias e resoluções, encadeando os seus raciocínios de forma organizada e coerente. As dificuldades são parte integrante deste processo, uma vez que é algo que

https://doi.org/10.48489/quadrante.33950
Received: December 2023 / Accepted: July 2024

pode ser sentido pelos alunos no momento que resolvem problemas ou escrevem as suas resoluções, e é necessário conseguir identificar essas dificuldades para posteriormente as ultrapassar. Posto isto, procuramos alcançar os seguintes objetivos: (1) compreender como os alunos comunicam as suas resoluções de problemas por escrito e (2) identificar quais as dificuldades sentidas por eles na resolução de problemas e na comunicação escrita. Desenvolveu-se um sistema de categorias para caracterizar a comunicação escrita e para analisar as dificuldades observadas nas resoluções dos alunos. As resoluções analisadas foram escritas por seis grupos de alunos do 11.º ano, a um problema matemático que lhes foi proposto. Obtivemos uma resolução incorreta, uma parcialmente correta e quatro corretas, com níveis de justificação a variar entre o nulo e o alto, e com todos os tipos de justificações passíveis de encontrar. Quanto às dificuldades, um dos grupos não apresentou qualquer dificuldade, enquanto outros tinham dificuldades variáveis entre a persistência, interpretação, seleção de informação, estratégia, escrita e coerência.
Palavras-chave: comunicação escrita; resolução de problemas; dificuldades; ensino secundário.

Introduction

In an era in which we are increasingly surrounded by technology, there is a need to take the best out of it and develop the skills that make us different. Problem-solving is one of them. So it is increasingly important to develop this skill and not just focus on memorising facts and rules, and applying algorithms (OECD, 2014). According to Højgaard (2021), problem-solving aims to document and/or display the process of attempting to solve a particular problem. In addition to problem-solving, communication skills are also crucial, especially because communication is “a structuring element of human activity”[1] (Menezes et al., 2013, p. 46, personal translation). The authors reinforce this importance by clarifying that we live in constant interaction with others, which means that many things we do involve communication.

In what is currently considered in Portugal as essential learnings for the discipline of Mathematics (Carvalho e Silva et al., 2023), problem-solving and written communication are two of the nine key ideas presented. This document states that problem-solving should be used to make connections between different concepts, either in Mathematics or in other areas of knowledge. Furthermore, students should be encouraged to justify their resolutions by properly linking their ideas and reasoning processes, using appropriate representations. Representations are directly related to mathematical communication, namely written communication, which is also very prominent in the above cited document. In addition to multiple representations, aspects related to the clarity of communication are mentioned, a point that we revisit in this paper. Finally, there is a focus on formative assessment, something that we hope this article can contribute to, as we defined categories of analysis for written resolutions of mathematical problems, which could prove to be very useful for formative evaluation of problem-solving in the classroom.

Errors and difficulties are common in problem-solving and written communication. It is important to be aware of errors made throughout the resolutions of a problem, rather than just focus on whether the answer is correct or not (Socas, 2007). Identifying the error makes it possible to work on it, helping to understand where it came from and avoid repeating the same error. Nevertheless, attention to errors must be moderate (Mascle, 2013), as we must also value the positive evolution that a student struggles for in order to remain motivated. By overcoming difficulties, this development takes place both in problem-solving and in written communication. For this reason, it is important to be able to predict the difficulties that students may experience. In this respect, Schoenfeld (1985) states that it is necessary to know in advance what kind of difficulties may arise so that, building in this knowledge, the teacher can bring more and better tools to help the students. It is necessary to help them to identify and overcome these difficulties so that the student continues to have the opportunity to solve the proposed task independently (Stein et al., 1996). These authors recommend helping the students in such a way that keeps them focused on the task and their ideas, without leading to excessive help. Menezes et al. (2014) states that, in this way, the support that the teacher gives to a student is focused on overcoming the difficulties that exist at that moment, allowing them to move forward on their own from then on. That is, without the teacher showing students the way to the answer they are expected to find.

This study focuses on written communication and difficulties in solving mathematical problems by secondary school students. Its main objectives are (1) *Understanding how students communicate their problem-solving in writing* and (2) *Identifying the difficulties they experience in problem-solving and written communication*. To achieve these objectives, we intend to present the results of the analysis of solutions to a proposed problem, written by groups of 11th grade students from two Science and Technology classes .

Written communication in mathematical problem-solving

Communication is the process of transmitting information between two or more people to achieve a certain goal (Thao & Trinh, 2018). Thus, communication is a form of social interaction that requires the people involved to adapt in order to build interpersonal relationships, exchange information and ideas, or encourage certain attitudes (Goma et al., 2020; Yuniara et al., 2018). As a socialisation process, communication allows us to live in a community and share our knowledge and thoughts (Menezes & Nacarato, 2020), as well as to develop argumentation, discussion, and reflection skills, as well as acceptance of other people's ideas. This interaction, according to Sáenz-Ludlow and Kadunz (2016), is mediated through the use of signals that are sent and subsequently interpreted, creating a cycle of communication between people and jointly leading to the creation of meaning. In terms of mathematical communication, it may involve the use of specific vocabularies with mathematical terminologies, or of rules that can substantiate possible arguments used (Thao & Trinh, 2018). Moschkovich (2002; 2018) states that there are differences between the language used in everyday life and the mathematical language, and these differences can represent obstacles when communicating mathematically. However, this author warns that everyday language can be useful for a better understanding of mathematical concepts, which reinforces that this difference in languages need not hinder communication. Boavida et al. (2008) point out that one of the main characteristics of mathematical language is its rigour and precision. Perhaps obstacles appear because mathematical language leaves no room for ambiguity, necessitating a thorough understanding of mathematical concepts for effective communication. But with or without a more specific vocabulary, according to Cross (2009), language is crucial for the development of mathematical competence.

Mathematical writing can be an intellectually demanding activity (Pantaleon et al., 2018) for students to express their reasoning in their own words, with images or mathematical models (Yuniara et al., 2018). Furthermore, it is important to note that written communication accompanies students throughout their entire school career, without being much noticed, whether in taking notes, making calculations or solving problems (Lee et al., 2020). It is a main tool to interact with others, and to record our ways of thinking and the development and evolution of our ideas (McCarthy, 2010). Therefore, writing can help students as well as the teachers who accompany them. According to Martinho and Rocha (2018), writing can help students to develop the right intuitions about mathematics and develop their ability to argue. On the other hand, Pugalee (2001) states that it can help teachers to have a better perception of how their students learn and how they think mathematically.

It is important to help students to practice their mathematical writing in order to improve the writing of their resolutions so that they become increasingly clear and complete (Martinho & Rocha, 2018). To achieve this, students are exposed to "tasks that encourage students to develop their written communication skills in mathematics, recording their ideas in a clear, correct and logical way"[2] (Costa & Pires, 2016, p. 407, personal translation). These authors also add that these records must be made using different representations and with a suitable justification of their reasoning. According to Aineamani (2018), the fact that a student knows how to explain their thinking with appropriate justifications means that the student has understood the knowledge underlying that task. The author emphasizes the importance of students having opportunities to justify their reasoning. Ponte and Quaresma (2020) assert that for this to occur, the classroom environment must encourage such engagement.

In order to understand how students communicate their resolutions in writing, we need to establish categories of analysis to characterise them. A possible system of categories of analysis is recommended by Santos and Semana (2015), based on three items: the interpretation of the task, the justification presented, and the representations used. Concerning *interpretation*, the authors consider that we should analyse whether the student correctly identified the objective and data in a task and what language was used. *Justification* requires the analyses of not only the argument's correctness and completeness, but also the type of justification used. There are four types of justification: vague (very brief and uninformative justification), rules (use of mathematical formulas, rules or definitions), procedural description (explanation of what is done in a given step) and relational justification (when explaining the validity of a step, which may or may not include an explanation of what is done in that step). Finally, the *representations* used – verbal language, iconic representation or symbolic representation – are analysed as well as the precision and completeness of the representation. Martinho and Rocha (2018) adapted these categories and defined a new list with four classes, some of them similar to those of Santos and Semana (2015), namely understanding the problem, presenting the resolution, justifying the answer, and explaining the representations used. The classes *understanding of the problem* and the *representations used* are defined as in Santos and Semana (2015). The *presentation of the resolution* class focuses on whether the steps taken in the resolution were explicit or not, and how justifications were presented. Finally, the *justification of the response* class emphasizes the analyses of the level and type of justification. The level of justification considers the correctness of the response, its clarity and completeness, while when analysing the type of justification, one considers the categories already presented by the previous authors: vague, rules, procedural, and relational, with an additional category added – the use of experimentation. Corrêa (2013), which refers to another dimension of analysis, related to the presentation of the response, or the lack of it. Regarding the lack of response, this author points out that the response may be absent from the resolution, but still be presented implicitly.

Based on this model, Martins (2023) presents four points of analysis: correction, completeness, representations and organisation. In *correction*, the authors assume that the answer may be correct, partially correct or incorrect. If the answer presented is correct, then this is exactly the assignment given in the correction category. For a partially correct answer, there are two possibilities – it can be partially correct concluded if the solution is not completely correct but presents significant correct parts, or partially correct not concluded if the resolution is not complete but what is written is significantly correct. An answer is incorrect as long as most of it is. In terms of *completeness*, this category is divided into three subcategories: level of justification, type of justification, and final answer. The level of justification can be high, medium, low or null, depending on the number of justifications given. The type of justification can be relational, procedural, resort to experimentation, exclusive use of rules, or vague. However, within the completeness category, we have the final answer, which is considered to be explicit (when an explicit answer to the problem is presented), implicit (in cases where the answer is not explicitly presented but it is possible to infer what the final answer was), and absent (when it is neither explicit nor implicit, or when more than one answer is presented). The third category of analysis focuses on the *representations* mentioned above, namely verbal language, iconic representation, or symbolic representation. Finally, for the *organization* category, Martins (2023) identifies three levels: organised, partially organised, and disorganised. In the first level, the organised response, we assume that the resolution is well organised and has a common thread that allows us to follow it from beginning to end without any problems. Being partially organised means it includes parts that require the reader to make connections between different steps to understand which one should be read first. Disorganisation occurs when the reader is prevented from properly following the resolution presented. This was the model we used to carry out the analysis of the results presented below.

Difficulties in problem-solving and written communication

Predicting the mistakes students may make and the difficulties they may experience is crucial. Mora and Rodriguez (2020) wonder whether errors and difficulties are the same thing. These authors note that some studies use these terms as synonyms, although this may be a controversial position. According to Mora and Rodriguez (2020), errors can occur for several reasons, including lack of knowledge, or simply by distraction. Difficulties arise from more complex networks that undergo transformations – a difficulty can become an obstacle that is later expressed by the student in the form of an error. For these authors, such difficulties can be related to mathematical knowledge, the complexity of mathematical objects, teaching processes and the cognitive development of the student.

Errors are often the result of difficulties experienced by students. Students may have different difficulties, so the teaching and learning process will not be the same for everyone. One way to help students is to understand their difficulties so that the acquisition of mathematical skills does not deteriorate over time (Tambychik & Meerah, 2010). These authors categorises difficulties based on specific abilities, listing five possibilities: numerical facts (related to difficulties in mathematical knowledge in general), arithmetic (failures in the precision of calculations or in the application of algorithms), information (difficulties in interpreting the problem statement), language (related to certain vocabulary known by the student that can act an obstacle) and spatial visualisation (difficulties in visualising more abstract concepts). In their study, Tambychik and Meerah (2010) confirm that, in many situations, students have difficulties in solving problems due to the lack of development of certain mathematical skills. Difficulties related to the information capacity is the most observed in this study. Something similar is also observed in the study by Ponte et al. (2012), finding out that students have significant difficulties in understanding the statement, especially when it is exposed in a less familiar context. With similar categories of analysis, Prates et al. (2011) choose to divide the difficulty of interpretation into three distinct topics: symbolism, information given, and interpretation of the solution. Symbolic difficulties arise when students do not know how to identify mathematical symbols, such as the domain and range symbols. The difficulty in the information given stands for the sorts of difficulties arising in collecting information from the statement and its use throughout the resolution of the problem. Finally, there may still be difficulties in interpreting solutions if the student is unable to provide an answer that combines what is requered and what has been developed in the resolution.

Wijaya et al. (2014) present a classification of difficulties similar to the one mentioned above, enriched with other two: unknown and transformation. An *unknown* type of difficulty arises when one finds a resolution where it is clear that there was some difficulty, this is still not sufficiently visible in the work submitted by the student. The difficulty of *transformation* occurs when students immediately use some knowledge they have learned, namely formulas or algorithms, without first reflecting on whether they are necessary. This difficulty also includes cases where too much attention is paid to the real context of the problem without looking at the underlying mathematical structure. Kusumadewi and Retnawati (2020) identify a type of difficulty related to the definition of strategy. This difficulty arises when the student is unable to define a strategy to solve the problem or uses inappropriate strategies. These authors also add the difficulty of facing the problem, which relates to persistence. This difficulty often starts with the fact that students are faced with large statements, with a lot of text (Phonapichat et al. 2014). For these authors, students do not like problems with a lot of text. Furthermore, they are more likely to cause comprehension difficulties, which does not help to properly solve the proposed problem.

For Martinho and Rocha (2018), students' failure to write their resolutions may not be due to a lack of mathematical knowledge, but rather to their inability to write down their ideas. These authors identify specific difficulties, such as translating reasoning into writing, knowing how to express their ideas, and how to structure the resolution so that the ideas are presented in a coherent and properly linked way. Morgan et al. (2014) also mentions these difficulties in the chaining of ideas and the establishment of logical connections that would need to be made, adding difficulties related to vocabulary and algebraic notation, as well as difficulties related to the length of texts. Thus, there are difficulties related to the writing process itself, but also to the phase of thinking about how to write and connect different stages of resolution. Yuniara et al. (2018) indicate three points where difficulties may be visible: concepts, procedures, and patience. The first one arises when students are unable to explain their ideas in writing. *Procedural* difficulties occur when they are not able to express a certain situation through drawings, mathematical models or even already known formulas. Finally, the authors present difficulties related to the principle of *patience*, where students are unable to express the situation through a mathematical model and are unable to apply previously known formulas or rules. Martins and Martinho (2021) also add the difficulty related to *coherence*, which occurs when students are not coherent throughout their solution, starting with initial assumptions and using contradictory arguments in the development of their reasoning.

In this research, we adopt a system of data analysis categories to classify the difficulties observed in students' written resolutions (Martins, 2023; Martins & Martinho, 2021). This system has nine main categories: persistence, interpretation, selection and organisation of information, strategy, process, writing, coherence, unknown, and none. Starting with persistence, it can be observed in two phases: beginning and conclusion. The difficulty of *persistence at the beginning* is when the student does not start to solve the problem and gives up without presenting any kind of resolution. *Persistence in conclusion* occurs when the student starts to solve the problem but is unable to complete it and gives up on solving it. The *interpretation* difficulty occurs in the *statement* – when the student does not understand what is said in the statement –, in *diagrams* – when the student does not correctly understand the information contained in a diagram, figure or scheme –, or in the *result* – when the student does not correctly interpret their resolution and the result they reached. The *selection and organisation of information* difficulty arises when *collecting the information* given in the statement, or even in the information acquired during its resolution, or in the *translation* between the verbal language and the mathematical language. The fourth category relates to *strategy*, including difficulties in *choosing the strategy* to apply to the proposed problem, or even in *executing* the chosen strategy. Within the *process* category, four are identified: concepts, arithmetic, rules, and generalisation. Thus, the student may have difficulties with *mathematical concepts*, with *calculations carried out* (arithmetic), in the *use and application of previously known mathematical rules*, or with the *generalisation or particularisation* of information given or acquired. Difficulties within the *writing* category can be divided into difficulties of conversion, structuring and connection. In other words, the student may have difficulties in *converting* their reasoning into writing, in *structuring* the writing of their resolution, or in *connecting* different ideas and stages of a resolution. Finally, the difficulty related to *coherence* arises when the student presents inconsistencies at different stages of their resolution. The categories *unknown* and *none* are intended for cases where the student's difficulty is not sufficiently visible in the work presented or when the student does not present any type of difficulty, respectively.

Context and methodology

This study involved the participation of 29 students from two 11th grade classes of the Science and Technology course, aged between 15 and 16. They attended a public school in an urban centre in the district of Braga and had the same mathematics teacher, both in the current and the previous school year. The ProbleMath.Com project was introduced to the students in these two classes, and it was explained that it would involve problem-solving sessions in groups, conducted online outside of school hours. All members of both classes were invited to register. One of the classes had 21 students and the other 24, with 14 and 15 students, respectively, taking part in the project on a volunteer basis. Students were divided into six working groups, three from each class. These groups were organised by the students themselves and remained fixed for all project sessions.

Sixteen 90-minute sessions, coordinated by the first author of this paper, took place between October 2020 and May 2021 and were recorded in full. In these sessions, students entered a common room and were then sent to separate rooms, one group in each room. At this stage, the researcher sent the problem proposed for that session to all students, who were given a maximum time to solve the problem in group. Each group was expected to solve the proposed problem and to collaboratively produce a written resolution that was as complete and well-justified as possible. At the end, all groups sent a file with their resolution to the researcher, e.g., in the form of a photo of the notebook, and met again in the common room, where space was open for a discussion of each group's resolutions. Everyone was invited to present their resolutions, in a certain order defined by the researcher, and all groups were encouraged to explain their ideas and to question their colleagues' resolutions. The order in which the resolutions were presented was determined by the researcher to make the discussion more interesting and relevant for the students, to show different resolution strategies, different ways of presenting the resolutions and to discuss the difficulties that arose. In this article, we present the analysis of the written communication of the resolutions that the groups developed for one of the problems proposed during one of these sessions and identify the observed difficulties.

In this research, it is assumed that our way of seeing the real world depends on personal interpretations developed according to our culture and past experiences. For this reason, this research takes a nominalist position (Neuman, 2014). Furthermore, it is part of an interpretative paradigm, as it is an investigation characterised by a strong interaction between the researcher and the persons under investigation (Coutinho, 2011). This interaction between the researcher and the students who make up the case under study was present in all stages of data collection, and the results obtained are highly dependent on the interpretations made by the researcher (Leavy, 2017). For this reason, the results are self-contained and cannot be generalised, although it is possible to draw conclusions that can be used in other cases (Neuman, 2014). This study is also part of a qualitative methodology, in which the data include the voices of the participants through the transcription of dialogues and their written resolutions, and a description of these data is made together with a reflection on the part of the researcher (Creswell & Poth, 2018). Finally, it can be considered as an instrumental case study since, according to Stake (2005), the case is only a means to achieve the phenomenon to be studied, and the case to be studied consists of students who volunteered to participate in this project.

To analyse the resolutions collected for this investigation, the following categories of analysis were applied to the written communication:

- Correction: analysing whether the solution presented is *correct*, *partially correct* (distinguishing whether this partially exists in a concluded or not concluded resolution), or *incorrect*
- Completeness:
    - Level of justification: analysing if the students presented all the necessary justifications in their resolution, distinguishing between *high*, *medium*, *low*, or *null*.
    - Type of justification: analysing whether the justifications presented are based on a *relational* understanding between different stages of the resolution; whether the justifications are about what is done in a particular stage (*procedural*); whether the justification is based on *experimentation* or based on *rules*; or whether it is simply a *vague* or uninformative justification.
    - Final answer: analysing whether the answer to the problem is presented *explicitly*, *implicitly*, or *absent.*
- Representations: analysing whether the students used *verbal language*, drawing or diagrams (*iconic representation*), or algebraic symbols (*symbolic representation*)
- Organization: analysing whether the answer is *organised*, with a common thread that allows the reader to follow the solution from beginning to end without problems, or whether it is only *partially organised* or even *disorganised*.

Regarding difficulties, the categories used were:

- Persistence: when the student is unable to start solving the problem (*beginning*) or to complete it (*conclusion*)
- Interpretation: when the student does not understand correctly what is written in the *statement*, or in *diagrams*, or does not interpret correctly their own *result*
- Selection and organization of information: if the student does not correctly *collect* the information given in the statement or presents difficulties in *translating* between verbal language and mathematical language

- Strategy: if the student has difficulties in *choosing* or *executing* the strategy to use to solve the problem
- Process: if the student has difficulties with mathematical *concepts*, *arithmetic*, *rules*, or in *generalising* or particularising information given or acquired
- Writing: if the student has difficulties in *converting* their reasoning into writing, or in *structuring* the resolution, or in *connecting* different ideas and stages of the resolution
- Coherence: if the student shows inconsistencies in their solution
- Unknown: when it is obvious that there was a difficulty, but the resolution presented is not sufficient for us to understand what that difficulty was
- None: if the student does not present any difficulties.

After summarizing the points of analysis, we move on to presenting the problem proposed to the students and the respective resolutions.

Results

In this section, we present the analysis of the resolutions provided by the six groups of students to the following problem:

> At the beginning, each of the three players placed the coins they had brought in front of them.
> Then they started a certain game. Each time one of them lost, they had to double the number of coins of their opponent.
> The session ended when the loser could no longer pay, or when one of them had the same number of coins as at the beginning.
> Inês now has 7 coins, and her opponents have 16 and 28.
> Ângela started with 20 coins.
> How many games have they played? How many coins does Diana have at the moment? How many more games can they play?[3]

In solving this problem, only one of the groups, Group 2, gave an incorrect answer. Furthermore, it was the only resolution that had a null level of justification. Actually, this group did not start the discussion, nor did they present a resolution at that stage, as it was assumed that it would not significantly contribute to the debate. Figure 1 shows the resolution of this group.

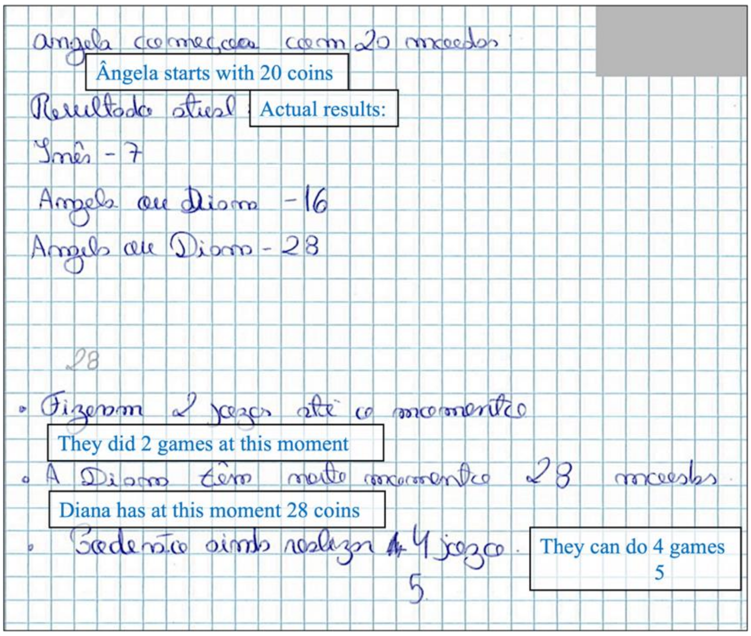

Figure 1. Problem resolution by Group 2

In the first four lines of the resolution, the group presents the collection of data for the problem. In the last three lines, the students include the answers to the three questions posed. Therefore, we assume that there is no strategy involved in this resolution, much less any type of argumentation. Furthermore, the answer is considered incorrect, as the three answers to the problem should be: they have played five games so far, Diana has 16 coins, and they can still play three more games. As the resolution presented in Figure 1 only has the answer to the problem, the level of justification is considered to be *null* and the justification is *vague*, even though the final answer is *explicit*. We have considered the use of *verbal language* in an *organised* resolution.

When it comes to difficulties, we immediately realise that they exist within the category of *persistence*. This is because the group does not begin solving the problem but merely writes down data from the statement and suggest a possible answer, so we could almost assume that there is a “non-answer”. Thus, we encountered difficulty in *persistence at the beginning*. This was due to another difficulty that became very apparent when taking field notes based on reviewing the session’s recordings: difficulty in *interpreting the statement*.

Sandra:	I don't understand... because it says that every time someone lost, they had to double the number of each opponent. But does this person go without, or does he just duplicate those of others?
Marta:	I think she runs out of coins, she will give each one twice of the coins, so she runs out…
Sandra:	So, let me see... (Group 2, session field note)

Almost 15 minutes later, Sandra manages to conclude that "[Ângela] has 20... pretend that Inês is x and Diana is y. So, at 20 we will have to take $x \times 2$ and $y \times 2$..." (Sandra, session field note). However, this was not enough to solve the problem, which is why it was difficult to *choose a strategy*.

We now move on to Group 5. They were the first group to present a resolution in the final discussion of this session, which happened to be an incomplete resolution. Therefore, the resolution presented in Figure 2 served as the starting point for a discussion in which all groups could complete the missing argumentation.

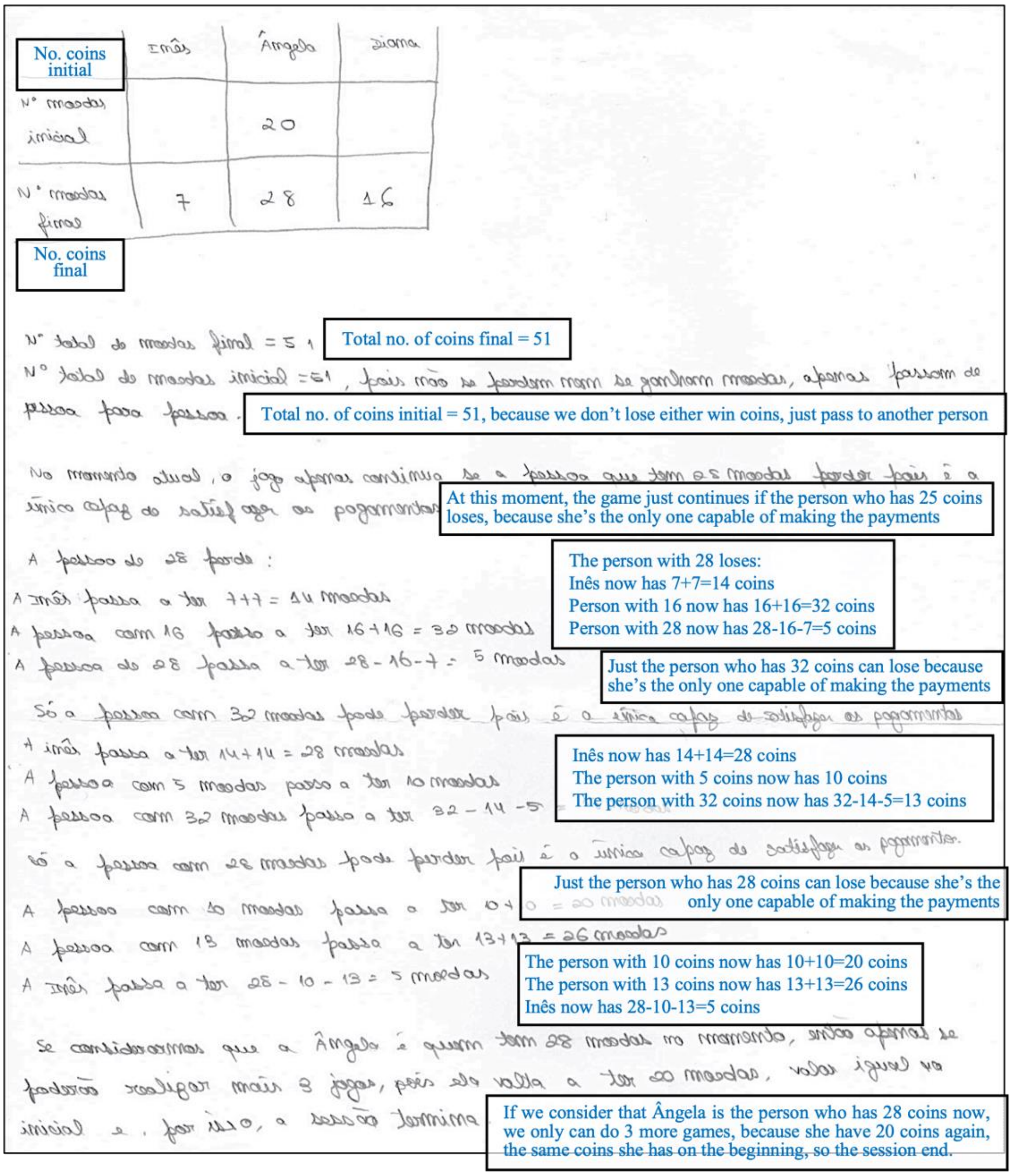

Figure 2. Problem resolution by Group 5

Since this group did not answer all the questions in the problem, but their solutions were correct, we consider their resolution to be *partially correct (not concluded)*, with a *high* level of justification and an *explicit* answer. For the types of justification, we considered the *procedural* type, since they justify what they are doing at each stage (for example, why they think that there are always 51 coins in each round), and *relational*, which is visible in the round in question. In terms of representations, we find *verbal language* and *iconic representation*, the latter translated into tabular form, and the resolution is *organised*. Regarding difficulties, the group showed difficulties in *interpreting the statement*, which was confirmed by watching the recording of the session, as it happened with the previous group. This reports they felt the need to call the researcher to clarify the problem.

Cintia:	One thing I still haven't understood is: they bring 30 coins each, imagine. Do they bet 30 coins on each game?
Edgar:	I think so...
Cintia:	But that would make the game end soon...
Clara:	But I don't think that's what it's about... her betting. I think it's... what does it mean to lose? Does that mean you lost a coin? (Group 5, session field notes)

After some discussion and asking for help to confirm that they were interpreting the problem correctly, they proceeded with the resolution and overcame this difficulty. However, the combination of the recording and the presented resolution also revealed a difficulty: the difficulty in *collecting information*, as they did not realise that two questions remained to be answered.

After Group 5 presented its resolution, a member of Group 6 volunteered to present his group's reasoning for answering two questions that remained unanswered. Their resolution is shown in Figure 3.

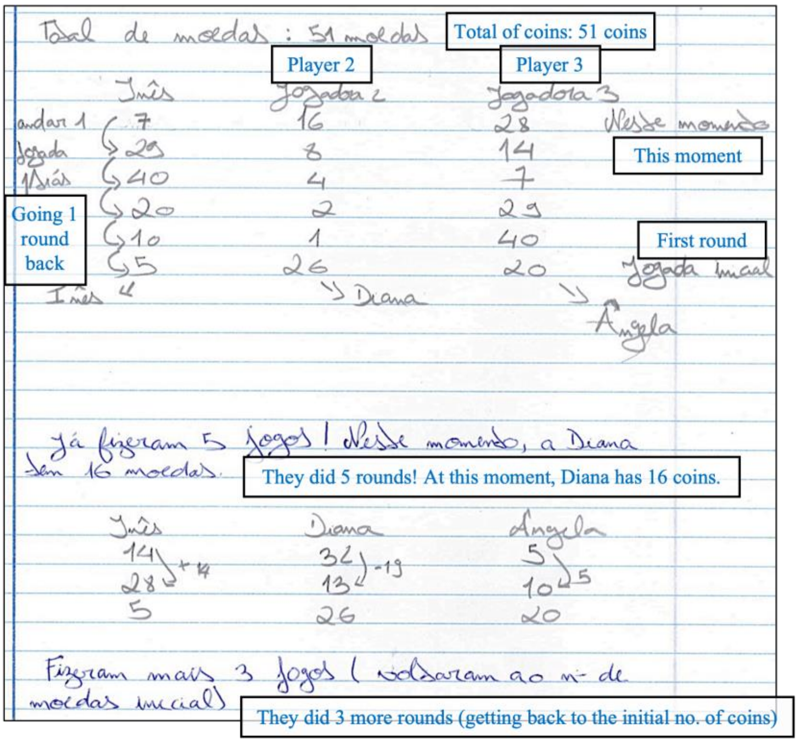

Figure 3. Problem resolution by Group 6

This group gave a *correct* answer to all the questions. However, the level of justification is *low*, as they do not explain why they thought that the first move would be Inês with 5 coins, Diana with 26 and Ângela with 20 (this is in the first part of the resolution). Just as they don't explain why, in the second half, they assume that it will be Ângela who will lose in the next round, they don't even make it clear that it was this player who lost. This absence makes the resolution *partially organised* since the reader needs to go back to the beginning of the first scheme to interpret why Inês starts the second scheme with 14 coins. In the absence of any other justification beyond the attempt presented in the schemes, we only consider the existence of *experimentation*, just as we only rely on the presence of *iconic representation* (since the verbal language used is not considered relevant). Therefore, it is considered that in this resolution, the group has shown difficulties in writing in terms of *structuring* the resolution and of the *connection* between the different stages of it.

Bearing in mind the intention to conclude the discussion with Group 1, the next explanation should come from one of the two remaining groups: Group 3 or Grupo 4. These groups were in an equal position, as they presented correct resolutions, with only slight differences Figure 4 presents the solution developed by Group 3.

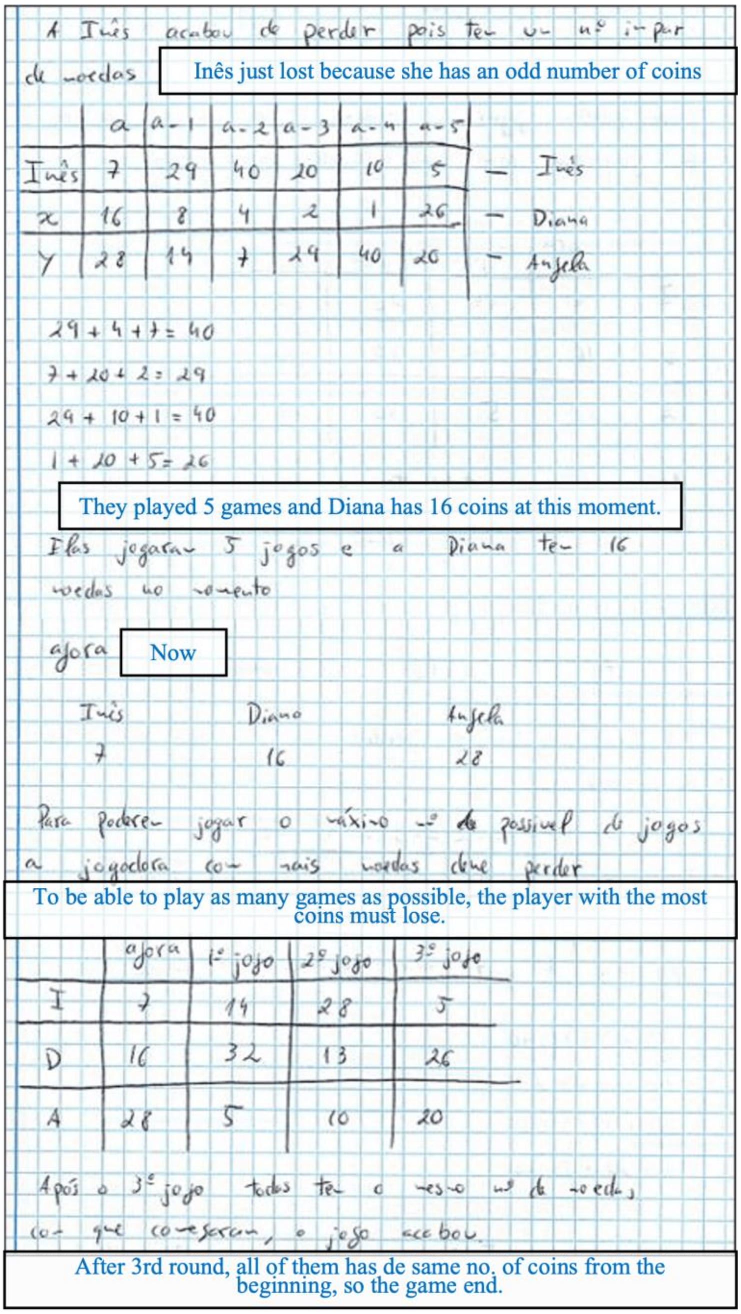

Figure 4. Problem resolution by Group 3

The answers given by Group 3 to the three questions are *explicit*, and the students used the *three types of representation* to write their resolutions in an *organized* presentation. The level of justification is *medium*. To reach the highest level, they would need to justify why they immediately assumed in "a-5" that Ângela was the friend with 28 coins at the current moment in the game and why they said that Inês would be the next to lose, just by having an odd number of coins. The type of justification is classified as *rules* because many of the steps are justified based on algorithms, mainly on the tables themselves, and *relational* when they justify, for example, that the person with the most coins must lose so that they have the greatest number of moves possible. In this case, the only difficulty recorded is at the connection level, reflecting what has already been indicated for the medium justification level.

Group 4 presents a similar resolution, a part of which can be seen in Figure 5.

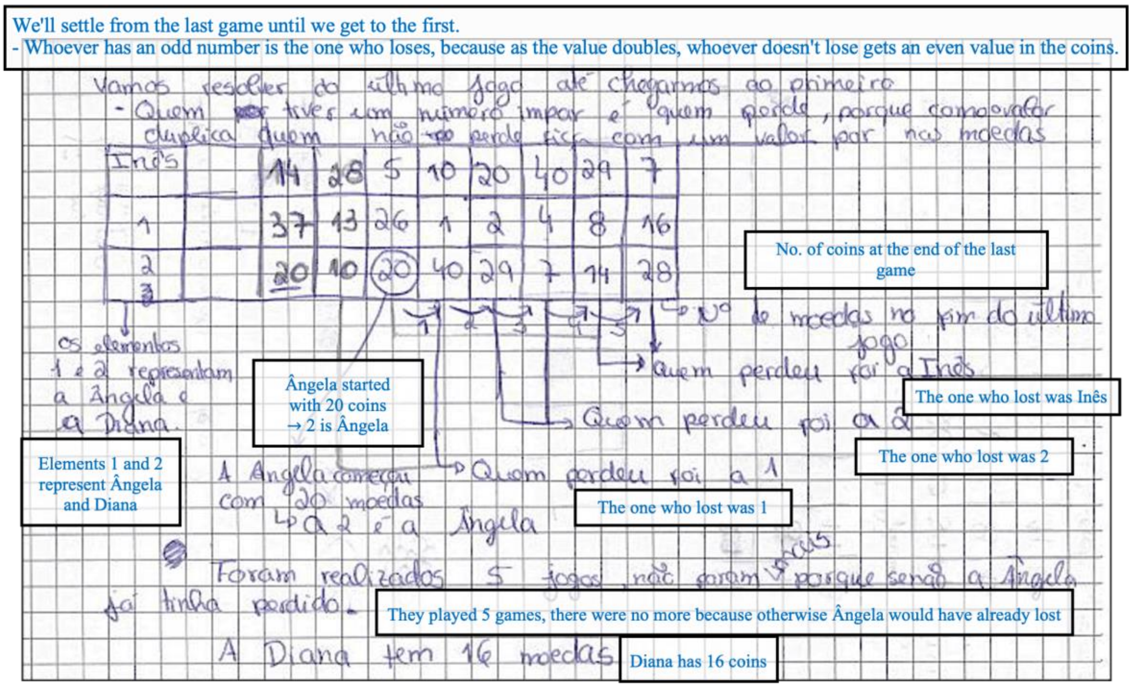

Figure 5. Part of problem resolution by Group 4

The resolution of Group 4 was very similar to that of Group 3, as mentioned above. For this reason, this article only includes the part of Group 4's resolution that differs from that of Group 3. These differences begin with the level of justification, which is *high*. This group presents all the justifications considered necessary. Another difference is in the type of justification: the *relational* type still occurs, but together with the *procedural*, since they justify what they do in each round of the game. In addition, this group resorts to *verbal language* and *iconic representation*, and unlike Group 3, it does not have the three types of representation. Finally, as far as difficulties are concerned, only coherence is taken into account due to columns three and four of the table presented, where values 14, 37, 20 and 28, 13, 10, do not seem to make sense in the context of the rest of the resolution. Possibly, these columns remained in the resolution of Group 4 by negligence or typing errors.

Finally, the discussion ended with Group 1, as it presented a *correct* and distinct resolution from the other groups. Figure 6 shows the first part of that resolution.

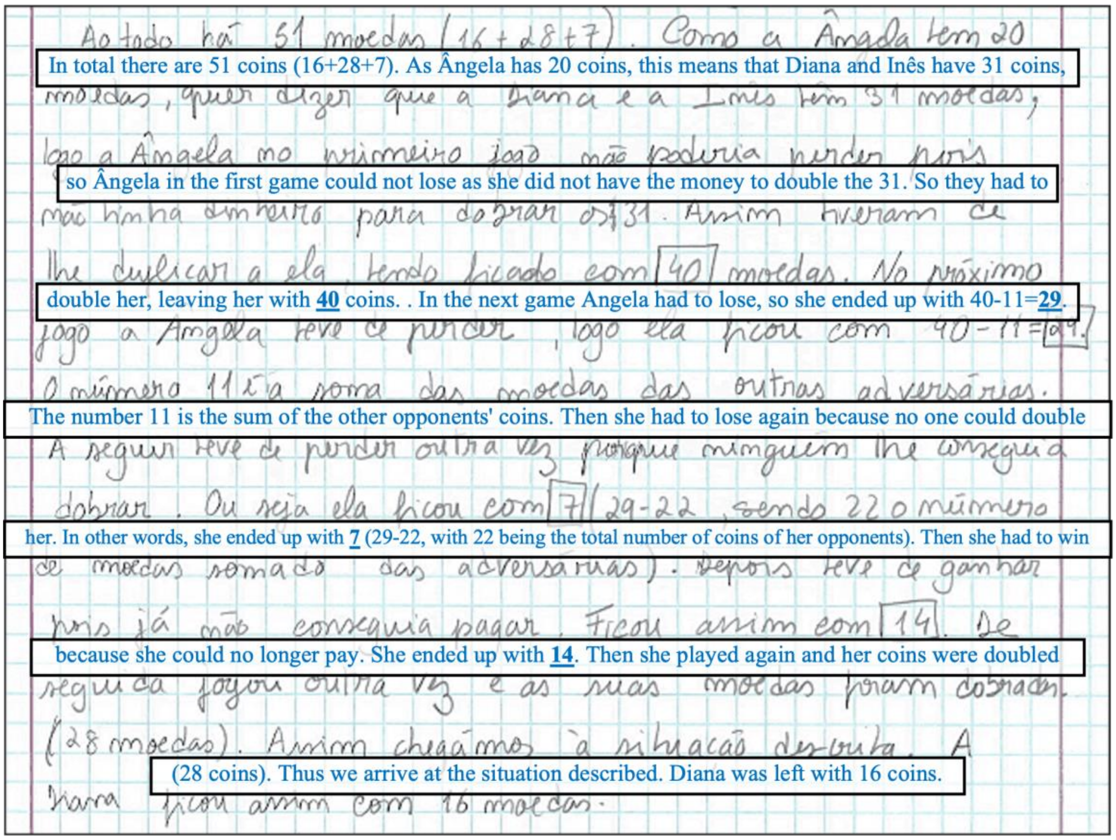

Figure 6. Part 1 of problem resolution by Group 1

This group used the information that Ângela started with 20 coins as a starting point and deduced from there. Figure 6 displays the first part of the Group 1's solution, which answers the question of how many coins Diana currently has. The continuation in shown in Figure 7. In this second part, the answer to the question "How many games have you played?" is first presented. Then, the second part of the solution addresses the last question posed in the problem. At the end, the group carried out a verification with the coins that Inês had at the beginning of the game.

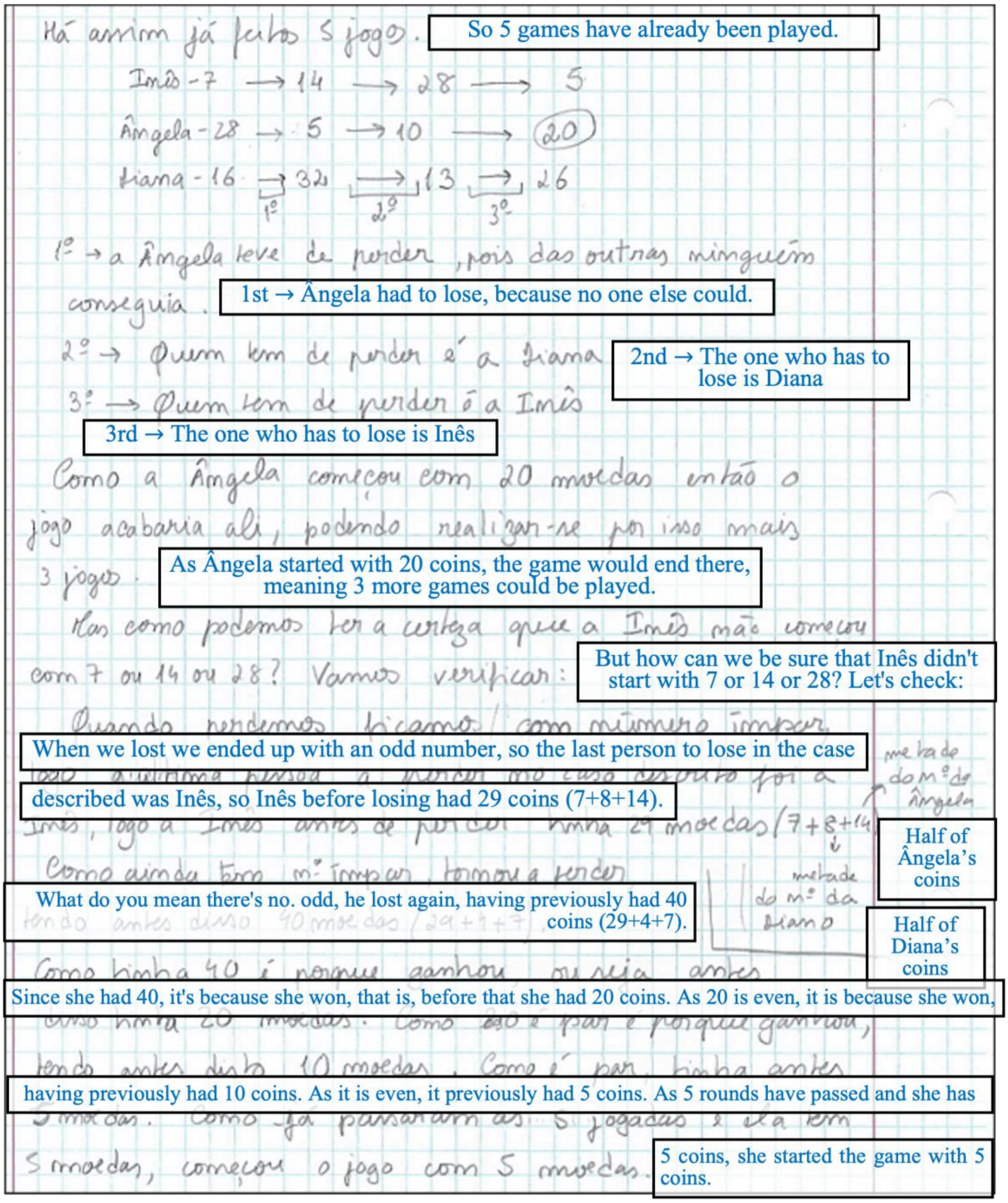

Figure 7. Part 2 of problem resolution by Group 1

This is a *correct*, *explicit* and *organised* resolution, with a *high* level of justification. The resolution of Group 1 essentially uses *verbal language*, although in the scheme presented they also resort to an *iconic representation*. All steps are duly justified, both in terms of what is done and why it is done, encompassing *procedural* and *relational* types of justification, respectively. Regarding difficulties, *none* were found.

In Table 1, we summarize the findings of what was mentioned in each of the analyses in this section.

Table 1. Summary table analyzing written communication and difficulties observed in the session

Written communication			G1	G2	G3	G4	G5	G6
Correction			C	I	C	C	PC-N	C
Completeness	Level of justification		H	N	M	H	H	L
	Type of justification	Relational	X	-	X	X	X	-
		Procedural	X	-	-	X	X	-
		Experimentation	-	-	-	-	-	X
		Rules	-	-	X	-	-	-
		Vague	-	X	-	-	-	-
	Final answer		E	E	E	E	E	E
Representations	Verbal language		X	X	X	X	X	-
	Iconic representation		X	-	X	X	X	X
	Symbolic representation		-	-	X	-	-	-
Organisation			O	O	O	PO	O	PO
	Difficulties		**G1**	**G2**	**G3**	**G4**	**G5**	**G6**
Persistence	Beginning		-	X	-	-	-	-
Interpretation	Statement		-	X	-	-	X	-
Selection and organization of information	Collecting		-	-	-	-	X	-
Strategy	Choosing		-	X	-	-	-	-
Writing	Structuring		-	-	-	-	-	X
	Connecting		-	-	X	-	-	X
Coherence			-	-	-	X	-	-
None			X	-	-	-	-	-

Source: Personal production.
Note. The symbol "X" means that this aspect is present in the resolution of that group. Furthermore: "C" correct; "PC-N" partially correct not concluded; "I" incorrect; "H" high; "M" medium; "L" low; "N" null; "E" explicit; "O" organized; "PO" partially organized

As we see at the previous table, four groups have correct resolutions, and in the remaining two one resolution that was not completed, and another was incorrect – although it could almost be argued that this group did not solve the problem. At the justification level, we find all the levels of our category system: null, low, medium and high. Medium- and high-level resolutions are those resorting to types of relational justification, while the high level also combined the procedural type. Only one group used symbolic representation and almost all used verbal language combined with some sort of iconic representation (namely using diagrams or tables). In terms of difficulties, one of the groups had no difficulties and another had a small problem related to coherence. Two groups had difficulties related to the problem interpretation, and the other two groups did not reach the correct answer to the problem. One of them also had difficulties in collecting information, and another revealed difficulty concerning persistence and in choosing a strategy.

Final considerations

Problem-solving should be used to establish connections between different concepts and to foster written communication, aligning with the guidelines outlined in the Essential Learning for Mathematics A (Carvalho e Silva et al., 2023). Difficulties are something that students naturally encounter when solving problems and writing their ideas. Socas (2007) warned that it is important not to focus only on the fact that the answer is correct, but better understand the mistakes students make and the difficulties they experience. Therefore, in this study we try to combine these three aspects: problem-solving, difficulties, and written communication. To this end, two main objectives were established: (1) to understand how students communicate their problem-solving in writing, and (2) to identify the difficulties they experience in problem-solving and written communication. To achieve these goals, we created an environment that encouraged students to communicate their mathematical ideas, both orally and in writing, as suggested by Ponte and Quaresma (2020). In addition, we developed a system of categories for both written communication and difficulties in problem-solving and written communication, based on nomenclatures and corresponding terms from various authors.

Regarding written communication, in the six resolutions of a problem proposed to 11th grade students presented and analysed in this paper, we found one incorrect resolution, one partially correct but not completed, and the remaining four correct. At the justification level, all possible levels were observed, with one null, one low, one medium and the remaining three high. It is important to emphasise that the null level was verified in the incorrect answer, but the partially correct answer presented a high level of justification (we note that this resolution was only partially correct because of a difficulty in collecting the information from the statement). The three high-level resolutions presented types of relational and procedural justification, with the medium-level one also presenting procedural justification in conjunction with rules. The group with a low level of justification was based just on experimentation and was the only group to use iconic representation exclusively. The group with a null level presented a vague justification, using only verbal language. The remaining four groups used verbal language and iconic representation simultaneously, with one of them also using symbolic representation – but although it was the only group to use all three types of representation, it was only at the medium level of justification. In a way, this is not entirely consistent with Moschkovich (2018), who mentioned that an interaction between the three types of representation would be important to contribute to better writing. However, we see here that the only group that used all the representations did not have what could be considered a high level of justification, while three groups at a higher level did not use the three possible representations simultaneously in their resolutions.

In terms of difficulties, one of the groups with a high level of justification did not present any difficulties, and another presented only a minor difficulty in terms of coherence. The third group with this level of justification had difficulties in interpreting the statement and collecting data – because they did not give an answer to two of the three questions in the problem. Two of the groups with a correct answer had difficulties connecting different stages of writing the resolution and one of them also had difficulties in structuring this writing. The group with the incorrect resolution had more difficulties: first with the interpretation of the statement, then with choosing the strategy, culminating in the lack of persistence at the beginning of the resolution. As the problem statement is extensive, difficulties in interpretation may have occurred for this reason (Phonapichat et al., 2014). Author1 and Author2 (2021) mentioned that improving the reading of the statement and collecting the restricted information it contains, without tending to invent data that do not exist, helps to overcome interpretation difficulties. In one of the groups (Group 2), this difficulty persisted and led to difficulties in persistence, but in the other group (Group 5), this difficulty was eventually overcome, and they managed to solve part of the problem. This happened because the students in Group 5 were persistent and read the problem several times until they were able to extract the correct information from the problem and interpret it appropriately.

Acknowledgements

This research was funded by FCT – Fundação para a Ciência e a Tecnologia through a doctoral scholarship (SFRH/BD/147510/2019). This work was also funded by CIEd – Centro de Investigação em Educação, Instituto de Educação, Universidade do Minho, projects UIDB/01661/2020 and UIDP/01661/2020 and UID/CED/1661/2016, through national funds from FCT/MCTES-PT.

Notes

[1] Original quote: "um elemento estruturante da atividade humana"

[2] Original quote: "tarefas que apelem ao desenvolvimento das suas capacidades de comunicação escrita em matemática, registando as suas ideias de forma clara, correta e lógica".
[3] From Viana, J. P. (2005). *Desafios 9* [Problem 39]. Edições Afrontamento.

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